

# A FAMILY OF HYPERBOLIC SPIN CALOGERO-MOSER SYSTEMS AND THE SPIN TODA LATTICES

LUEN-CHAU LI

**ABSTRACT.** In this paper, we continue to develop a general scheme to study a broad class of integrable systems naturally associated with the coboundary dynamical Lie algebroids. In particular, we present a factorization method for solving the Hamiltonian flows. We also present two important class of new examples, a family of hyperbolic spin Calogero-Moser systems and the spin Toda lattices. To illustrate our factorization theory, we show how to solve these Hamiltonian systems explicitly.

## 1. Introduction.

In the theory of integrable systems, a wide range of important examples are covered by the Adler-Kostant-Symes scheme and its generalization known as classical r-matrix theory (see [A], [K], [S], [RSTS1], [RSTS2], [AvM], [STS1],[STS2], [RSTS3], [FT], [LP1] and the references therein). As is well-known, classical r-matrices are naturally associated with Poisson structures on Lie groups and duals of Lie algebras and the corresponding geometric objects have been used with great success in the solutions of many integrable Hamiltonian systems.

In the early 90's, dynamical analog of the classical r-matrices was discovered in the study of Wess-Zumino-Witten (WZN) conformal field theory [BDF], [F]. Since then, these objects have cropped up in other areas as well (see, for example, [BAB], [ABB], [Lu], [AM]) and their geometric meaning was unraveled by Etingof and Varchenko in their fundamental paper [EV]. While classical r-matrices play a role in Poisson Lie group theory [D], the authors in [EV] showed that an appropriate geometric setting for the classical dynamical r-matrices is that of a special class of Poisson groupoids (a notion due to Weinstein [W1]), the so-called coboundary dynamical Poisson groupoids. If  $R$  is an  $H$ -equivariant classical dynamical r-matrix, and  $(\Gamma, \{\cdot, \cdot\}_R)$  is the associated coboundary dynamical Poisson groupoid, then it follows from Weinstein's coisotropic calculus [W1] or otherwise that the Lie algebroid dual  $A^*\Gamma$  also has a natural Lie algebroid structure [LP2], [BKS]. We shall

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call  $A^*\Gamma$  the coboundary dynamical Lie algebroid associated to  $R$  and it is this class of Lie algebroids which we use in the study of integrable systems in [LX2] and in the present work.

Our purpose in this paper is twofold. First of all, we will continue to develop a general scheme (which we initiated in [LX2]) to study integrable systems based on realization in the dual bundles of coboundary dynamical Lie algebroids. To summarize, the class of invariant Hamiltonian systems which admits such a realization (for the genuinely dynamical case) has the following key features: (a) the systems are defined on a Hamiltonian  $H$ -space  $X$  with equivariant momentum map  $J$  and the Hamiltonians are the pull-back of natural invariant functions by an  $H$ -equivariant realization map, (b) the pullback of natural invariant functions do not Poisson commute everywhere on  $X$ , but they do so on a fiber  $J^{-1}(\mu)$  of the momentum map, (c) the reduced Hamiltonian systems on  $X_\mu = J^{-1}(\mu)/H_\mu$  ( $H_\mu$  is the isotropy subgroup at  $\mu$ ) admit a natural collection of Poisson commuting integrals. In this work, our main focus is on the case in which  $R$  is a solution of the modified dynamical Yang-Baxter equation (mDYBE). As we pointed out in the announcement [L1], the (mDYBE) is associated with a factorization problem on the trivial Lie groupoid  $\Gamma$ . By making use of the algebraic and geometric structures associated with (mDYBE) (which will be fully worked out here), we will develop an effective method to integrate the Hamiltonian flows on  $J^{-1}(\mu)$  (which parallels the one announced in [L1] for the groupoid framework) based on this factorization. Hence we can obtain the integrable flows on  $X_\mu$  by reduction.

Our second purpose in this work is to give two important class of new examples and to illustrate our factorization theory using these examples. Our first class of examples is a family of hyperbolic spin Calogero-Moser (CM) systems and their associated integrable models, corresponding to the solutions of (mDYBE) for pairs  $(\mathfrak{g}, \mathfrak{h})$  of Lie algebras, as classified in [EV]. Here,  $\mathfrak{g}$  is simple, and  $\mathfrak{h} \subset \mathfrak{g}$  is a Cartan subalgebra. As such, our systems are parametrized by subsets  $\pi'$  of a simple system of roots  $\pi$ . Note that in the special case where  $\pi' = \pi$ , our corresponding integrable model is isomorphic to the one in [R], and the  $sl(N)$  case has also appeared in [AB], [KBBT], for example (see Remark 5.5). The second class of examples was actually discovered when an appropriate scaling limit is applied to the hyperbolic spin CM systems (the ones which are not integrable). Remarkably, the obstruction to integrability dissolves in the scaling limit, leading to a family of integrable models which we will call the spin Toda lattices (again, these are parametrized by subsets  $\pi'$  of a simple system). As it turns out, the spin Toda lattices are systems which

admit realization in the dual bundle  $\mathbf{A}$  of the coboundary dynamical Lie algebroid  $\mathbf{A}^* \simeq T\mathfrak{h} \times \mathfrak{g}$  associated to the standard r-matrix and the reduction of these  $H$ -invariant systems lead to a family of Toda lattices parametrized by  $\pi'$ . So this gives us a first nontrivial example in which a constant r-matrix is relevant.

The paper is organized as follows. In Section 2, we derive an intrinsic expression for the Lie-Poisson structure on the dual bundle of a coboundary dynamical Lie algebroid which is important for subsequent developments of our program. We also give the complete set of equations for a natural class of invariant Hamiltonian systems. In Section 3, we reprove (essentially) Theorem 3.10 in [LX2] using an intrinsic point of view, without having to assume the existence of an  $H$ -equivariant map  $g : X \longrightarrow H$  (also we do not assume  $\mathfrak{h} = \text{Lie}(H)$  is Abelian). We also compute how the realization map evolves under our invariant Hamiltonian systems on  $X$  based on the development in Section 2. As the reader will see, the integrable flows on the reduced space  $X_\mu$  are actually realized on a Poisson quotient of a coisotropic submanifold of  $A\Gamma$ , which in some sense is the analog of the gauge group bundle in [L1]. In Section 4, we discuss the algebraic and geometric structures associated with (mDYBE), leading up to a factorization method for solving the Hamiltonian flows. In Section 5, we introduce a family of hyperbolic spin Calogero-Moser systems using Proposition 4.2 (a) and consider the associated integrable models. Then we consider scaling limits of the hyperbolic spin CM systems at the levels of the Hamiltonians, the equations of motion and the (generalized) Lax equations. At the end of the section, we work out the realization picture for the spin Toda lattices and also consider their reduction. Section 6 is concerned with the solution of the hyperbolic spin CM systems and the spin Toda lattices, utilizing the factorization method of Section 4. Here, the reader will see how the concrete factorization problems are being solved. In a remark, we will also discuss the solution of a family of hyperbolic spin Ruijsenaars-Schneider models (introduced in [L1] and related to the affine Toda field theories [BH]) in the general case. We shall address the complete integrability and other aspects of the integrable models here in subsequent publications. For the solution of the systems in [LX2] using the method developed here, we refer the reader to the forthcoming work [L2] (see also Remark 5.5 (c)).

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## 2. Coboundary dynamical Lie algebroids and Lie-Poisson structures on their dual bundles.

In this section, our main goal is to derive an intrinsic formula for the Lie-Poisson structure on the dual bundle of a coboundary dynamical Lie algebroid which is important for subsequent developments. As the reader will see, the same method of calculation is also used in Section 5 to write down the Lie-Poisson structure associated with a trivial Lie algebroid, whose vertex Lie algebra is given by a semi-direct product.

We begin by recalling the definition of a Lie algebroid.

**Definition 2.1.** A Lie algebroid over a manifold  $M$  is a smooth vector bundle  $\pi_A : A \longrightarrow M$  equipped with a Lie bracket  $[\cdot, \cdot]_A$  on its space  $Sect(M, A)$  of smooth sections and a bundle map  $a_A : A \longrightarrow TM$  (called the anchor map) such that

- (a) the bundle map  $a_A$  induces a Lie algebra homomorphism  $Sect(M, A) \longrightarrow Sect(TM)$  (which we also denote by  $a_A$ ),
- (b) for any  $X, Y \in Sect(M, A)$  and  $f \in C^\infty(M)$ , the Leibnitz identity

$$[X, fY]_A = f[X, Y]_A + (a(X)f)Y$$

holds.

Let  $(A, [\cdot, \cdot]_A, a_A)$  be a Lie algebroid over  $M$ , and let  $\pi_{A^*} : A^* \longrightarrow M$  be its dual bundle. For any  $X \in Sect(M, A)$ , we can associate a smooth function  $l_X$  on  $A^*$  by putting  $l_X(\xi) = \langle \xi, X \circ \pi_{A^*}(\xi) \rangle$  for all  $\xi \in A^*$ .

**Theorem 2.2** [CDW]. *There exists a unique Poisson structure on  $A^*$  (called the Lie-Poisson structure) which is characterized by the property*

$$\{l_X, l_Y\} = l_{[X, Y]_A}$$

for all  $X, Y \in Sect(M, A)$ .

**Remark 2.3** In [C] and [W2], there are two extra conditions in the characterization of the Lie-Poisson structure, namely,  $\{f \circ \pi_{A^*}, g \circ \pi_{A^*}\} = 0$ , and  $\{l_X, f \circ \pi_{A^*}\} = (a(X)f) \circ \pi_{A^*}$  for all  $f, g \in C^\infty(M)$ , and  $X \in Sect(M, A)$ . However, it is not hard to show that these are actually consequences of  $\{l_X, l_Y\} = l_{[X, Y]_A}$  and the properties of the Lie algebroid bracket  $[\cdot, \cdot]_A$ . We shall use these two conditions in an essential way in (2.10) below.

We now recall the definition of a coboundary dynamical Lie algebroid. Let  $G$  be a connected Lie group, and  $H \subset G$  a connected Lie subgroup. We shall denote by  $\mathfrak{g}$  and  $\mathfrak{h}$  the corresponding Lie algebras and let  $\iota : \mathfrak{h} \longrightarrow \mathfrak{g}$  be the Lie inclusion. Let  $U \subset \mathfrak{h}^*$  be a connected  $Ad_H^*$ -invariant open subset, and let  $R : U \longrightarrow L(\mathfrak{g}^*, \mathfrak{g})$  be a classical dynamical r-matrix (here and henceforth we denote by  $L(\mathfrak{g}^*, \mathfrak{g})$  the set of linear maps from  $\mathfrak{g}^*$  to  $\mathfrak{g}$ ), i.e.  $R$  is pointwise skew symmetric

$$\langle R(q)(A), B \rangle = - \langle A, R(q)B \rangle \quad (2.1)$$

and satisfies the classical dynamical Yang-Baxter condition

$$\begin{aligned} & [R(q)A, R(q)B] + R(q)(ad_{R(q)A}^* B - ad_{R(q)B}^* A) \\ & + dR(q)\iota^* A(B) - dR(q)\iota^* B(A) + d \langle R(A), B \rangle (q) = \chi(A, B), \end{aligned} \quad (2.2)$$

for all  $q \in U$ , and all  $A, B \in \mathfrak{g}^*$ , where  $\chi : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}$  is  $G$ -equivariant.

The dynamical  $r$ -matrix is said to be  $H$ -equivariant if and only if

$$R(Ad_h^* q) = Ad_h \circ R(q) \circ Ad_h^* \quad (2.3)$$

for all  $h \in H, q \in U$ . We shall equip  $\Gamma = U \times G \times U$  with the trivial Lie groupoid structure over  $U$  [M] with target and source maps

$$\alpha(u, g, v) = u, \quad \beta(u, g, v) = v \quad (2.4)$$

and multiplication map

$$m((u, g, v), (v, g', w)) = (u, gg', w). \quad (2.5)$$

Recall that associated with an  $H$ -equivariant classical dynamical r-matrix  $R$  is a natural Poisson structure  $\{\cdot, \cdot\}_R$  on  $\Gamma$  [EV] such that the pair  $(\Gamma, \{\cdot, \cdot\}_R)$  is a Poisson groupoid in the sense of Weinstein [W1] (see [L1] for the intrinsic forms of  $\{\cdot, \cdot\}_R$ ). Let  $A\Gamma := \bigcup_{q \in U} T_{\epsilon(q)} \alpha^{-1}(q) = \bigcup_{q \in U} \{0_q\} \times \mathfrak{g} \times \mathfrak{h}^*$  be the Lie algebroid of  $\Gamma$ . Then by Weinstein's coisotropic calculus [W1] or otherwise, the Lie algebroid dual  $A^*\Gamma = \bigcup_{q \in U} \{0_q\} \times \mathfrak{g}^* \times \mathfrak{h}$  also has a natural Lie algebroid structure [BKS], [LP2] such that the pair  $(A\Gamma, A^*\Gamma)$  is a Lie bialgebroid in the sense of Mackenzie and Xu [MX]. We shall denote the Lie brackets on  $Sect(U, A\Gamma)$  and  $Sect(U, A^*\Gamma)$  respectively by  $[\cdot, \cdot]_{A\Gamma}$  and  $[\cdot, \cdot]_{A^*\Gamma}$ . Throughout the paper, the pair  $(A^*\Gamma, [\cdot, \cdot]_{A^*})$  together with the anchor map  $a_* : A^*\Gamma \longrightarrow TU$  given by

$$a_*(0_q, A, Z) = (q, \iota^* A - ad_Z^* q) \quad (2.6)$$

will be called the coboundary dynamical Lie algebroid associated to  $R$ . Explicitly, the Lie bracket  $[\cdot, \cdot]_{A^*\Gamma}$  on  $Sect(U, A^*\Gamma)$  is given by the following expression [BKS],[LP2]:

$$\begin{aligned}
& [(0, A, Z), (0, A', Z')]_{A^*\Gamma}(q) \\
&= (0_q, dA'(q)(\iota^* A(q) - ad_{Z(q)}^* q) - dA(q)(\iota^* A'(q) - ad_{Z'(q)}^* q) \\
&\quad - ad_{R(q)A(q)-Z(q)}^* A'(q) + ad_{R(q)A'(q)-Z'(q)}^* A(q), \\
&\quad dZ'(q)(\iota^* A(q) - ad_{Z(q)}^* q) - dZ(q)(\iota^* A'(q) - ad_{Z'(q)}^* q) \\
&\quad - [Z, Z'](q) + \langle dR(q)(\cdot)A(q), A'(q) \rangle)
\end{aligned} \tag{2.7}$$

where  $A, A' : U \longrightarrow \mathfrak{g}^*$ ,  $Z, Z' : U \longrightarrow \mathfrak{h}$  are smooth maps and  $\langle dR(q)(\cdot)A(q), A'(q) \rangle$  is the element in  $\mathfrak{h}$  whose pairing with  $\lambda \in \mathfrak{h}^*$  is  $\langle dR(q)(\lambda)A(q), A'(q) \rangle$ .

In the rest of the section, we shall make the identifications

$$A\Gamma \simeq U \times \mathfrak{h}^* \times \mathfrak{g}, \quad A^*\Gamma \simeq U \times \mathfrak{h} \times \mathfrak{g}^*. \tag{2.8}$$

Let us fix a point  $(q, \lambda, X) \in A\Gamma$ . In order to derive an intrinsic expression for the Lie-Poisson bracket  $\{\varphi, \psi\}_{A\Gamma}(q, \lambda, X)$  on the dual bundle  $A\Gamma$  of the coboundary dynamical Lie algebroid  $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma}, a_*)$ , we need to introduce some notation. To start with, let  $Pr_i$  be the projection map onto the  $i$ -th factor of  $U \times \mathfrak{h}^* \times \mathfrak{g} \simeq A\Gamma$ ,  $i = 1, 2, 3$ . If  $\varphi \in C^\infty(A\Gamma)$ , we have  $d\varphi(q, \lambda, X) = (\delta_1\varphi, \delta_2\varphi, \delta\varphi)$ , where the partial derivatives are defined by

$$\begin{aligned}
\langle \delta_1\varphi, \mu \rangle &= \frac{d}{dt}\bigg|_{t=0} \varphi(q + t\mu, \lambda, X), \quad \langle \delta_2\varphi, \mu \rangle = \frac{d}{dt}\bigg|_{t=0} \varphi(q, \lambda + t\mu, X), \quad \mu \in \mathfrak{h}^* \\
\langle \delta\varphi, Y \rangle &= \frac{d}{dt}\bigg|_{t=0} \varphi(q, \lambda, X + tY), \quad Y \in \mathfrak{g}.
\end{aligned}$$

We also associate with  $\varphi$  the function  $\tilde{\varphi}$  on  $U$ , defined by  $\tilde{\varphi}(u) = \varphi(u, \lambda, X)$ . On the other hand,  $s(\varphi) : U \longrightarrow U \times \mathfrak{h} \times \mathfrak{g}^*$  will denote the constant section of  $U \times \mathfrak{h} \times \mathfrak{g}^*$  given by  $s(\varphi)(u) = (u, \delta_2\varphi, \delta\varphi)$ , where  $\delta_2\varphi, \delta\varphi$  are the partial derivatives evaluated at the fixed point  $(q, \lambda, X)$ .

Now, it is easy to check by a direct calculation that  $d(\tilde{\varphi} \circ Pr_1)(q, \lambda, X) = (\delta_1\varphi, 0, 0)$ , while  $dl_{s(\varphi)}(q, \lambda, X) = (0, \delta_2\varphi, \delta\varphi)$ . Thus we have

$$d\varphi(q, \lambda, X) = d(l_{s(\varphi)} + \tilde{\varphi} \circ Pr_1)(q, \lambda, X). \tag{2.9}$$

Therefore,

$$\begin{aligned}
& \{\varphi, \psi\}_{A\Gamma}(q, \lambda, X) \\
&= \{l_{s(\varphi)} + \tilde{\varphi} \circ Pr_1, l_{s(\psi)} + \tilde{\psi} \circ Pr_1\}_{A\Gamma}(q, \lambda, X) \\
&= l_{[s(\varphi), s(\psi)]_{A^*\Gamma}}(q, \lambda, X) + d\tilde{\psi}(q)a_*(s(\varphi))(q) \\
&\quad - d\tilde{\varphi}(q)a_*(s(\psi))(q).
\end{aligned} \tag{2.10}$$

By using the expression for  $[\cdot, \cdot]_{A^*\Gamma}$  in (2.7), we have

$$\begin{aligned} & l_{[s(\varphi), s(\psi)]_{A^*\Gamma}}(q, \lambda, X) \\ &= \langle \lambda, -[\delta_2\varphi, \delta_2\psi] + \langle dR(q)(\cdot)\delta\varphi, \delta\psi \rangle \rangle \\ & \quad + \langle X, -ad_{R(q)\delta\varphi - \delta_2\varphi}^*\delta\psi + ad_{R(q)\delta\psi - \delta_2\psi}^*\delta\varphi \rangle. \end{aligned}$$

Meanwhile, from the expression for the anchor map  $a_*$ , we find

$$d\tilde{\varphi}(q)a_*(s(\varphi))(q) = \langle \delta_1\varphi, \iota^*\delta\psi - ad_{\delta_2\psi}^*q \rangle.$$

Assembling the calculations, we have the following result.

**Theorem 2.4.** *The Lie-Poisson structure on the dual bundle  $A\Gamma$  of the coboundary dynamical Lie algebroid  $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma}, a_*)$  is given by*

$$\begin{aligned} & \{\varphi, \psi\}_{A\Gamma}(q, \lambda, X) \\ &= -\langle \lambda, [\delta_2\varphi, \delta_2\psi] \rangle + \langle dR(q)(\lambda)\delta\varphi, \delta\psi \rangle \\ & \quad + \langle X, -ad_{R(q)\delta\varphi - \delta_2\varphi}^*\delta\psi + ad_{R(q)\delta\psi - \delta_2\psi}^*\delta\varphi \rangle \\ & \quad - \langle q, [\delta_2\varphi, \delta_1\psi] + [\delta_1\varphi, \delta_2\psi] \rangle + \langle \delta_1\psi, \iota^*\delta\varphi \rangle - \langle \delta_1\varphi, \iota^*\delta\psi \rangle. \end{aligned}$$

**Remark 2.5** In a similar fashion, we can show that the Lie-Poisson bracket on the dual bundle  $A^*\Gamma$  of the trivial Lie algebroid  $(A\Gamma, [\cdot, \cdot]_{A\Gamma}, a)$  is given by  $\{\varphi, \psi\}_{A^*\Gamma}(q, p, \xi) = \langle \delta_2\varphi, \delta_1\psi \rangle - \langle \delta_1\varphi, \delta_2\psi \rangle + \langle \xi, [\delta\varphi, \delta\psi] \rangle$ . The reader is referred to Proposition 5.10 below for the details of a similar calculation.

If  $(P, \{\cdot, \cdot\}_P)$  is a Poisson manifold, then for each  $f \in C^\infty(P)$ , we shall define the associated Hamiltonian vector field  $X_f$  using the convention  $X_f.g = \{f, g\}_P$ .

**Corollary 2.6.** *The Hamiltonian vector field on  $A\Gamma$  associated to  $\varphi \in C^\infty(A\Gamma)$  is given by*

$$\begin{aligned} & X_\varphi(q, \lambda, X) \\ &= (\iota^*\delta\varphi - ad_{\delta_2\varphi}^*q, -ad_{\delta_2\varphi}^*\lambda + \iota^*ad_X^*\delta\varphi - ad_{\delta_1\varphi}^*q, \\ & \quad [X, R(q)\delta\varphi - \delta_2\varphi] + dR(q)(\lambda)\delta\varphi - \delta_1\varphi + R(q)(ad_X^*\delta\varphi)). \end{aligned}$$

Now, a natural collection of invariant functions on  $A\Gamma$  is  $Pr_3^*I(\mathfrak{g})$ , where  $I(\mathfrak{g})$  is the ring of ad-invariant functions on  $\mathfrak{g}$ . The following result is an easy consequence of Theorem 2.4 and Corollary 2.6.

**Corollary 2.7.** (a) *The Hamilton's equation on  $A\Gamma$  generated by  $Pr_3^*f$ ,  $f \in I(\mathfrak{g})$  is of the form*

$$\begin{aligned}\dot{q} &= \iota^* df(X), \\ \dot{\lambda} &= 0, \\ \dot{X} &= [X, R(q)df(X)] + dR(q)(\lambda)(df(X)).\end{aligned}$$

(b) *For all  $f_1, f_2 \in I(\mathfrak{g})$ , we have*

$$\begin{aligned}& \{Pr_3^*f_1, Pr_3^*f_2\}_{A\Gamma}(q, \lambda, X) \\ &= \langle dR(q)(\lambda)(df_1(X)), df_2(X) \rangle.\end{aligned}$$

**Remark 2.8** If  $R$  is a constant r-matrix, then it is immediate from Corollary 2.7 (b) above that functions in  $Pr_3^*I(\mathfrak{g})$  Poisson commute on  $A\Gamma$ . In this case, the equation for  $X$  in part (a) of the same corollary is a Lax equation in the standard r-matrix framework for Lie algebras [STS1]. So when  $R$  is constant, what we have here is a slight extension of the standard framework. For an example associated with a constant r-matrix which fits into our framework, we refer the reader to Section 5.

### 3. Realization of Hamiltonian systems in coboundary dynamical Lie algebroids.

Let  $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma}, a_*)$  be the coboundary dynamical Lie algebroid corresponding to  $R$ , and let  $\rho : X \longrightarrow A\Gamma$  be a realization of a Poisson manifold  $(X, \{\cdot, \cdot\}_X)$  in the dual bundle  $A\Gamma$  of the Lie algebroid  $A^*\Gamma$ , i.e.,  $\rho$  is a Poisson map. If  $Pr_i$  is the projection map onto the  $i$ -th factor of  $U \times \mathfrak{h}^* \times \mathfrak{g} \simeq A\Gamma$ ,  $i = 1, 2, 3$ , we put

$$m = Pr_1 \circ \rho : X \longrightarrow U, \quad (3.1)$$

$$\tau = Pr_2 \circ \rho : X \longrightarrow \mathfrak{h}^*, \quad (3.2)$$

$$L = Pr_3 \circ \rho : X \longrightarrow \mathfrak{g}. \quad (3.3)$$

We shall make the following assumptions:

- A1.  $X$  is a Hamiltonian  $H$ -space with an equivariant momentum map  $J : X \longrightarrow \mathfrak{h}^*$ ,
- A2. the realization map  $\rho$  is  $H$ -equivariant, where  $H$  acts on  $A\Gamma$  via the formula

$$h \cdot (q, \lambda, X) = (Ad_{h^{-1}}^* q, Ad_h^* \lambda, Ad_h X), \quad (3.4)$$

- A3. for some regular value  $\mu \in \mathfrak{h}^*$  of  $J$ ,

$$\rho(J^{-1}(\mu)) \subset U \times \{0\} \times \mathfrak{g}. \quad (3.5)$$

Note that the condition in (3.4) is the natural generalization of the corresponding condition in [LX2] since we do not assume  $\mathfrak{h}$  is Abelian here. On the other hand, our assumption A3 is stronger than what we had in [LX2]. Our purpose in this section is to exhibit the intrinsic role played by the orbit space  $(U \times \{0\} \times \mathfrak{g})/H$  of the action in (3.4) in the reduction to integrable flows. We also compute how the realization map evolves under our invariant Hamiltonian systems on  $X$ .

**Proposition 3.1.** *With the action defined in (3.4), the dual bundle  $A\Gamma$  of the coboundary dynamical Lie algebroid  $A^*\Gamma$  equipped with the Lie-Poisson structure is a Hamiltonian  $H$ -space with equivariant momentum map  $\gamma : A\Gamma \longrightarrow \mathfrak{h}^*$ ,  $(q, \lambda, X) \mapsto \lambda$ .*

*Proof.* Denote the action by  $\Phi$ . If  $\varphi \in C^\infty(A\Gamma)$ , it follows by a direct calculation that  $\delta_i(\varphi \circ \Phi_h)(q, \lambda, X) = Ad_{h^{-1}}\delta_i\varphi(\Phi_h(q, \lambda, X))$ ,  $i = 1, 2$  and  $\delta(\varphi \circ \Phi_h)(q, \lambda, X) = Ad_h^*\delta\varphi(\Phi_h(q, \lambda, X))$ . The assertion that  $\Phi_h$  is Poisson then follows upon using the formula in Theorem 2.4 and the fact that  $R$  is  $H$ -equivariant. Now, for any  $Z \in \mathfrak{h}$ , we have

$$\frac{d}{dt}\bigg|_{t=0} \Phi_{e^{tZ}}(q, \lambda, X) = (-ad_Z^*q, -ad_Z^*\lambda, ad_ZX).$$

Comparing the right hand side of the above expression with the formula in Corollary 2.6, it is clear that this is equal to  $X_{\hat{\gamma}(Z)}(q, \lambda, X)$ , where  $\hat{\gamma}(Z)(q, \lambda, X) = \langle \lambda, Z \rangle$ . Hence  $\gamma(q, \lambda, X) = \lambda$ .  $\square$

From this result, it follows that  $X = A\Gamma$ , and  $\rho = id_{A\Gamma}$  satisfy assumptions A1-A3 above with  $\mu = 0$  and we have  $\gamma^{-1}(0) = U \times \{0\} \times \mathfrak{g}$ .

We shall denote by  $H_\mu$  the isotropy subgroup of  $\mu$  for the  $H$ -action on  $X$ . Then it follows by Poisson reduction [MR], [OR] (see [OR] for the singular case) that the variety  $X_\mu = J^{-1}(\mu)/H_\mu$  inherits a unique Poisson structure  $\{\cdot, \cdot\}_{X_\mu}$  satisfying

$$\pi_\mu^*\{f_1, f_2\}_{X_\mu} = i_\mu^*\{\tilde{f}_1, \tilde{f}_2\}_X. \quad (3.6)$$

Here,  $i_\mu : J^{-1}(\mu) \longrightarrow X$  is the inclusion map,  $\pi_\mu : J^{-1}(\mu) \longrightarrow X_\mu$  is the canonical projection,  $f_1, f_2 \in C^\infty(X_\mu)$ , and  $\tilde{f}_1, \tilde{f}_2$  are (locally defined) smooth extensions of  $\pi_\mu^*f_1, \pi_\mu^*f_2$  with differentials vanishing on the tangent spaces of the  $H$ -orbits. For the case where  $X = A\Gamma$ ,  $\rho = id_{A\Gamma}$ , it is clear that the isotropy subgroup at  $\mu = 0$  is  $H$  itself and so we have the Poisson variety

$$(A\Gamma_0 = \gamma^{-1}(0)/H, \{\cdot, \cdot\}_{A\Gamma_0}), \quad (3.7)$$

with the inclusion map  $i_H : \gamma^{-1}(0) \longrightarrow A\Gamma$  and the canonical projection  $\pi_H : \gamma^{-1}(0) \longrightarrow A\Gamma_0$ .

Clearly, functions in  $i_H^* Pr_3^* I(\mathfrak{g}) \subset C^\infty(\gamma^{-1}(0))$  are  $H$ -invariant, hence they descend to functions in  $C^\infty(A\Gamma_0)$ . On the other hand, it follows from assumption A2 that the functions in  $i_\mu^* L^* I(\mathfrak{g}) \subset C^\infty(J^{-1}(\mu))$  drop down to functions in  $C^\infty(X_\mu)$ . Now, by assumption A2-A3, and the fact that  $\rho$  is Poisson, it follows from [OR] that  $\rho$  induces a unique Poisson map

$$\widehat{\rho} : X_\mu \longrightarrow A\Gamma_0 = (U \times \{0\} \times \mathfrak{g})/H \quad (3.8)$$

characterized by  $\pi_H \circ \rho \circ i_\mu = \widehat{\rho} \circ \pi_\mu$ . Hence  $X_\mu$  admits a realization in the Poisson variety  $A\Gamma_0$ .

We shall use the following notation. For  $f \in I(\mathfrak{g})$ , the unique function in  $C^\infty(A\Gamma_0)$  determined by  $i_H^* Pr_3^* f$  will be denoted by  $\bar{f}$ ; while the unique function in  $C^\infty(X_\mu)$  determined by  $i_\mu^* L^* f$  will be denoted by  $\mathcal{F}_\mu$ . From the definitions, we have

$$\mathcal{F}_\mu \circ \pi_\mu = (\widehat{\rho}^* \bar{f}) \circ \pi_\mu = i_\mu^* L^* f \quad (3.9)$$

**Theorem 3.2.** *Let  $(X, \{\cdot, \cdot\}_X)$  be a Poisson manifold which admits a realization  $\rho : X \longrightarrow A\Gamma$  and assume A1-A3 are satisfied. Then there exist a unique Poisson structure  $\{\cdot, \cdot\}_{X_\mu}$  on the reduced space  $X_\mu = J^{-1}(\mu)/H_\mu$  and a unique Poisson map  $\widehat{\rho}$  such that*

(a) *for all  $f_1, f_2 \in I(\mathfrak{g})$ ,  $x \in J^{-1}(\mu)$ , we have*

$$\begin{aligned} & \{\widehat{\rho}^* \bar{f}_1, \widehat{\rho}^* \bar{f}_2\}_{X_\mu} \circ \pi_\mu(x) \\ &= \langle L(x), -ad_{R(m(x))}^* df_1(L(x)) df_2(L(x)) + ad_{R(m(x))}^* df_2(L(x)) df_1(L(x)) \rangle. \end{aligned}$$

(b) *functions  $\widehat{\rho}^* \bar{f}$ ,  $f \in I(\mathfrak{g})$ , Poisson commute in  $(X_\mu, \{\cdot, \cdot\}_{X_\mu})$ ,*

(c) *if  $\psi_t$  is the induced flow on  $\gamma^{-1}(0) = U \times \{0\} \times \mathfrak{g}$  generated by the Hamiltonian  $Pr_3^* f$ ,  $f \in I(\mathfrak{g})$ , and  $\phi_t$  is the Hamiltonian flow of  $\mathcal{F} = L^* f$  on  $X$ , then under the*

flow  $\phi_t$ , we have

$$\begin{aligned}\frac{d}{dt}m(\phi_t) &= \iota^* df(L(\phi_t)), \\ \frac{d}{dt}\tau(\phi_t) &= 0, \\ \frac{d}{dt}L(\phi_t) &= [L(\phi_t), R(m(\phi_t))df(L(\phi_t))] + dR(m(\phi_t))(\tau(\phi_t))df(L(\phi_t))\end{aligned}$$

where the term involving  $dR$  drops out on  $J^{-1}(\mu)$ . Moreover, the reduction  $\phi_t^{red}$  of  $\phi_t \circ i_\mu$  on  $X_\mu$  defined by  $\phi_t^{red} \circ \pi_\mu = \pi_\mu \circ \phi_t \circ i_\mu$  is a Hamiltonian flow of  $\mathcal{F}_\mu = \widehat{\rho}^* \bar{f}$  and  $\widehat{\rho} \circ \phi_t^{red}(\pi_\mu(x)) = \pi_H \circ \psi_t(\rho(x))$ ,  $x \in J^{-1}(\mu)$ .

*Proof.* (a) Since  $\rho(J^{-1}(\mu)) \subset U \times \{0\} \times \mathfrak{g}$ , we have  $\tau(x) = 0$  for  $x \in J^{-1}(\mu)$ . Therefore,

$$\begin{aligned}& \{\widehat{\rho}^* \bar{f}_1, \widehat{\rho}^* \bar{f}_2\}_{X_\mu \circ \pi_\mu(x)} \\ &= \{\bar{f}_1, \bar{f}_2\}_{A\Gamma_0 \circ \pi_H(\rho(x))} \\ &= \{Pr_3^* f_1, Pr_3^* f_2\}_{A\Gamma(\rho(x))} \\ &= \langle L(x), -ad_{R(m(x))df_1(L(x))}^* df_2(L(x)) + ad_{R(m(x))df_2(L(x))}^* df_1(L(x)) \rangle\end{aligned}$$

where in the last step we have invoked the formula in Theorem 2.4 and the vanishing of  $\tau(x)$  for  $x \in J^{-1}(\mu)$ .

(b) This is clear from part (a).

(c) Since  $\rho$  is a Poisson map, we have  $\frac{d}{dt}\rho(\phi_t) = X_{f \circ Pr_3}(\rho(\phi_t))$  from which the equations follow on invoking Corollary 2.7. Finally, the assertion on  $\phi_t^{red}$  is basically a corollary of Theorem 2.16 of [OR] and the relation  $\rho \circ \phi_t \circ i_\mu = \psi_t \circ \rho \circ i_\mu$ .  $\square$

**Remark 3.3** In [LX2], we have only written down the equation for  $L$  under the Hamiltonian flow  $\phi_t$  (in the Abelian case). However, the full set of equations is important. See Section 4 and Section 6 below.

#### 4. Factorization problems on Lie groupoids and exact solvability.

We shall develop a factorization method to solve the (generalized) Lax equations in Corollary 2.7 (a) on the level set  $\gamma^{-1}(0)$  of the momentum map  $\gamma$ . For the first part of this section, we shall use  $A\Gamma = \bigcup_{q \in U} \{0_q\} \times \mathfrak{g} \times \mathfrak{h}^*$ ,  $A^*\Gamma = \bigcup_{q \in U} \{0_q\} \times \mathfrak{g}^* \times \mathfrak{h}$ , and when  $\mathfrak{g}$  has an ad-invariant non-degenerate pairing, we shall identify the Lie algebras with their duals.

As in [L1], we introduce the bundle map

$$\mathcal{R} : A^*\Gamma \longrightarrow A\Gamma, (0_q, A, Z) \mapsto (0_q, -\iota Z + R(q)A, \iota^* A - ad_{Z^*}^* q) \quad (4.1)$$

and call it the *r-matrix of the Lie algebroid*  $A^*\Gamma$ . Also, we assume  $R$  satisfies the modified dynamical Yang-Baxter equation (mDYBE):

$$\begin{aligned} & [R(q)A, R(q)B] + R(q)(ad_{R(q)A}^* B - ad_{R(q)B}^* A) \\ & + dR(q)\iota^* A(B) - dR(q)\iota^* B(A) + d\langle R(A), B \rangle(q) \\ & = -[K(A), K(B)] \end{aligned} \quad (4.2)$$

where  $K \in L(\mathfrak{g}^*, \mathfrak{g})$  is a nonzero symmetric map which satisfies  $ad_X \circ K + K \circ ad_X^* = 0$  for all  $X \in \mathfrak{g}$ , i.e,  $K$  is  $G$ -equivariant.

The next two results were announced in [L1]. We give details of the proof here.

**Lemma 4.1.** *If  $R$  satisfies (mDYBE), then the r-matrix  $\mathcal{R} : A^*\Gamma \longrightarrow A\Gamma$  satisfies the equation*

$$\begin{aligned} & [\mathcal{R}(0, A, Z), \mathcal{R}(0, A', Z')]_{A\Gamma} - \mathcal{R}[(0, A, Z), (0, A', Z')]_{A^*\Gamma} \\ & = (0, -[K(A), K(A')], 0) \end{aligned} \quad (4.3)$$

for all smooth maps  $A, A' : U \longrightarrow \mathfrak{g}^*$ ,  $Z, Z' : U \longrightarrow \mathfrak{h}$ .

*Proof.* The calculation will be postponed to the appendix.  $\square$

Using  $K$ , we define

$$\mathcal{K} : A^*\Gamma \longrightarrow A\Gamma, (0_q, A, Z) \mapsto (0_q, K(A), 0), \quad (4.4)$$

and set  $\mathcal{R}^\pm = \mathcal{R} \pm \mathcal{K}$ ,  $R^\pm(q) = R(q) \pm K$ .

**Proposition 4.2.**

(a)  $\mathcal{R}^\pm$  are morphisms of transitive Lie algebroids and, as morphisms of vector bundles over  $U$ , are of locally constant rank. In particular,

$$[\mathcal{R}^\pm(0, A, Z), \mathcal{R}^\pm(0, A', Z')]_{A\Gamma} = \mathcal{R}^\pm[(0, A, Z), (0, A', Z')]_{A^*\Gamma} \quad (4.5)$$

for all smooth maps  $A, A' : U \longrightarrow \mathfrak{g}^*$ ,  $Z, Z' : U \longrightarrow \mathfrak{h}$ . Moreover,  $\mathcal{R}^\pm$  are  $H$ -equivariant, where  $H$  acts on  $A^*\Gamma$  via  $h \cdot (0_q, A, Z) = (0_{Ad_{h^{-1}}^* q}, Ad_{h^{-1}}^* A, Ad_h Z)$  and the  $H$ -action on  $A\Gamma$  is given by (3.4).

(b)  $Im\mathcal{R}^\pm$  are transitive Lie subalgebroids of  $A\Gamma$ .

*Proof.* (a) If  $a$  and  $a_*$  are the anchor maps of the Lie algebroids  $A\Gamma$  and  $A^*\Gamma$ , it is easy to check that they are surjective submersions which satisfy  $a \circ \mathcal{R}^\pm = a_*$ . On the other hand, it follows from Lemma 4.1 that (4.5) holds if and only if

$$\begin{aligned} & \mathcal{K}[(0, A, Z), (0, A', Z')]_{A^*\Gamma} \\ &= [\mathcal{R}(0, A, Z), \mathcal{K}(0, A', Z')]_{A\Gamma} + [\mathcal{K}(0, A, Z), \mathcal{R}(0, A', Z')]_{A\Gamma}. \end{aligned} \quad (4.6)$$

Now, for  $q \in U$ , we have

$$\begin{aligned} & \mathcal{K}[(0, A, Z), (0, A', Z')]_{A^*\Gamma}(q) \\ &= (0_q, K(dA'(q)(\iota^* A(q) - ad_{Z(q)}^* q) - ad_{R(q)A(q)-Z(q)}^* A'(q)) \\ & \quad - (A \leftrightarrow A', Z \leftrightarrow Z'), 0) \end{aligned}$$

where  $(A \leftrightarrow A', Z \leftrightarrow Z')$  denote terms which can be obtained from the previous ones by interchanging  $A$  and  $A'$ ,  $Z$  and  $Z'$ . On the other hand,

$$\begin{aligned} & [\mathcal{R}(0, A, Z), \mathcal{K}(0, A', Z')]_{A\Gamma}(q) \\ &= (0_q, K(dA'(q)(\iota^* A(q) - ad_{Z(q)}^* q)) + [-\iota Z(q) + R(q)A(q), K(A'(q))], 0) \end{aligned}$$

and similarly for  $-\mathcal{R}(0, A', Z'), \mathcal{K}(0, A, Z)]_{A\Gamma}(q)$ . From these formulas, it follows that (4.6) holds if and only if

$$\begin{aligned} & K(ad_{R(q)A(q)-Z(q)}^* A'(q)) + [-\iota Z(q) + R(q)A(q), K(A'(q))] \\ & - (A \leftrightarrow A', Z \leftrightarrow Z') = 0. \end{aligned}$$

But the latter follows from the  $G$ -equivariance of  $K$  and this proves the first part of the assertion. (The fact that  $\mathcal{R}^\pm$  are of locally constant rank follows from Theorem 1.6 on page 190 of [M].) To show that  $\mathcal{R}^\pm$  are  $H$ -equivariant, note that by definition,

$$\begin{aligned} & \mathcal{R}^\pm(h \cdot (0_q, A, Z)) \\ &= (0_{Ad_{h^{-1}}^* q}, -Ad_h Z + R^\pm(Ad_{h^{-1}}^* q)Ad_{h^{-1}}^* A, \iota^* Ad_{h^{-1}}^* A - ad_{Ad_h Z}^* Ad_{h^{-1}}^* q). \end{aligned}$$

But from the  $H$ -equivariance of  $R$  and  $K$ , we have  $R^\pm(Ad_{h^{-1}}^* q)Ad_{h^{-1}}^* A = Ad_h R^\pm(q)A$ . On the other hand, it is straightforward to check that  $ad_{Ad_h Z}^* Ad_{h^{-1}}^* q = Ad_{h^{-1}}^* ad_Z^* q$ . Substituting into the above expression for  $\mathcal{R}^\pm(h \cdot (0_q, A, Z))$ , the desired conclusion follows.

(b) This is a consequence of (a).  $\square$

In the rest of the section, we shall assume  $\mathfrak{g}$  has an ad-invariant non-degenerate pairing  $(\cdot, \cdot)$  such that  $(\cdot, \cdot)|_{\mathfrak{h} \times \mathfrak{h}}$  is also non-degenerate. Without loss of generality,

we shall take the map  $K : \mathfrak{g}^* \longrightarrow \mathfrak{g}$  in the above discussion to be the identification map induced by  $(\cdot, \cdot)$ . Indeed, with the identifications  $\mathfrak{g}^* \simeq \mathfrak{g}$ ,  $\mathfrak{h}^* \simeq \mathfrak{h}$ , we shall regard  $R(q)$  as taking values in  $End(\mathfrak{g})$ , and the left and right gradients as well as the dual maps are computed using  $(\cdot, \cdot)$ . Also, we have  $ad^* \simeq -ad$ ,  $\iota^* \simeq \Pi_{\mathfrak{h}}$ , where  $\Pi_{\mathfrak{h}}$  is the projection map to  $\mathfrak{h}$  relative to the direct sum decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ . We shall keep, however, the notation  $A^*\Gamma$  although as a set it can be identified with  $A\Gamma$ .

We now introduce the following subbundles of the adjoint bundle  $Ker a = \{(0_q, X, 0) \mid q \in U, X \in \mathfrak{g}\}$  of  $A\Gamma$ :

$$\mathcal{I}^+ = \{(0_q, X, 0) \in Ker a \mid q \in U, \mathcal{R}^-(0_q, X, Z) = 0 \text{ for some } Z \in \mathfrak{h}\}, \quad (4.7a)$$

$$\mathcal{I}^- = \{(0_q, X, 0) \in Ker a \mid q \in U, \mathcal{R}^+(0_q, X, Z) = 0 \text{ for some } Z \in \mathfrak{h}\}. \quad (4.7b)$$

**Proposition 4.3.**  $\mathcal{I}^\pm$  are ideals of the transitive Lie algebroids  $Im\mathcal{R}^\pm$ .

*Proof.* We shall prove the assertion for  $\mathcal{I}^+$ . First of all, it is easy to show that  $\mathcal{I}^+ \subset Im\mathcal{R}^+$ . Let  $(0, -\iota Z + R^+X, \Pi_{\mathfrak{h}}X + ad_Z(\cdot)) \in Sect(U, Im\mathcal{R}^+)$  and  $(0, X', 0) \in Sect(U, \mathcal{I}^+)$ , where  $Z : U \longrightarrow \mathfrak{h}$ ,  $X, X' : U \longrightarrow \mathfrak{g}$  are smooth maps. From the expression for  $[\cdot, \cdot]_{A\Gamma}$ , we have

$$\begin{aligned} & [(0, -\iota Z + R^+X, \Pi_{\mathfrak{h}}X + ad_Z(\cdot)), (0, X', 0)]_{A\Gamma}(q) \\ &= (0_q, dX'(q)(\Pi_{\mathfrak{h}}X(q) + ad_{Z(q)}q) + [-\iota Z(q) + R^+(q)X(q), X'(q)], 0) \end{aligned} \quad (4.8)$$

for  $q \in U$ . This shows

$$[(0, -\iota Z + R^+X, \Pi_{\mathfrak{h}}X + ad_Z(\cdot)), (0, X', 0)]_{A\Gamma} \in Sect(U, Ker a).$$

On the other hand, from the assumption that  $(0, X', 0) \in Sect(U, \mathcal{I}^+)$ , we must have  $\mathcal{R}^-(0_q, X'(q), Z'(q)) = 0$  for some smooth map  $Z' : U \longrightarrow \mathfrak{h}$ . Hence we obtain  $(0, X', 0) = \mathcal{R}^+(0, X'/2, Z'/2)$ . Therefore, on using Proposition 4.2 (a), it follows that

$$\begin{aligned} & [(0, -\iota Z + R^+X, \Pi_{\mathfrak{h}}X + ad_Z(\cdot)), (0, X', 0)]_{A\Gamma}(q) \\ &= \mathcal{R}^+[(0, X, Z), (0, X'/2, Z'/2)]_{A^*\Gamma}(q) \\ &= \mathcal{R}^+(0_q, X''(q), Z''(q)) \\ &= (0_q, -\iota Z''(q) + R^+(q)X''(q), \Pi_{\mathfrak{h}}X''(q) + [Z''(q), q]) \end{aligned} \quad (4.9)$$

where by (2.7), we find

$$\begin{aligned} X''(q) &= \frac{1}{2}dX'(q)(\Pi_{\mathfrak{h}}X(q) + ad_{Z(q)}q) + \frac{1}{2}[R(q)X(q) - Z(q), X'(q)] \\ &\quad + \frac{1}{2}[X(q), R(q)X'(q) - Z'(q)] \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} Z''(q) &= \frac{1}{2}dZ'(q)(\Pi_{\mathfrak{h}}X(q) + ad_{Z(q)}q) - \frac{1}{2}[Z, Z'](q) \\ &\quad + \frac{1}{2}(dR(q)(\cdot)X(q), X'(q)). \end{aligned} \quad (4.11)$$

By equating the last components of the expressions in (4.8) and (4.9), we have

$$\Pi_{\mathfrak{h}}X''(q) + [Z''(q), q] = 0. \quad (4.12)$$

Similarly, by equating the second components of the expressions in (4.8) and (4.9), we find

$$\begin{aligned} dX'(q)(\Pi_{\mathfrak{h}}X(q) + ad_{Z(q)}q) + [-\iota Z(q) + R^+(q)X(q), X'(q)] \\ = -\iota Z''(q) + R^+(q)X''(q) \end{aligned} \quad (4.13)$$

Now, from the relation  $\mathcal{R}^-(0_q, X'(q), Z'(q)) = 0$ , it follows that (4.10) can be rewritten as

$$\begin{aligned} dX'(q)(\Pi_{\mathfrak{h}}X(q) + ad_{Z(q)}q) + [-\iota Z(q) + R^+(q)X(q), X'(q)] \\ = 2X''(q). \end{aligned} \quad (4.14)$$

Substitute this into (4.12), we obtain

$$\begin{aligned} \Pi_{\mathfrak{h}}\{dX'(q)(\Pi_{\mathfrak{h}}X(q) + ad_{Z(q)}q) + [-\iota Z(q) + R^+(q)X(q), X'(q)]\} \\ + [2Z''(q), q] = 0. \end{aligned}$$

Next, (4.13) and (4.14) yield

$$\begin{aligned} R^-(q)\{dX'(q)(\Pi_{\mathfrak{h}}X(q) + ad_{Z(q)}q) + [-\iota Z(q) + R^+(q)X(q), X'(q)]\} \\ = 2Z''(q). \end{aligned}$$

From the last two relations, we can now conclude that

$$[(0, -\iota Z + R^+X, \Pi_{\mathfrak{h}}X + ad_Z(\cdot), (0, X', 0)]_{A\Gamma} \in Sect(U, \mathcal{I}^+),$$

as desired.  $\square$

Consider now the quotient vector bundles  $Im\mathcal{R}^{\pm}/\mathcal{I}^{\pm}$  equipped with the quotient transitive Lie algebroid structures.

**Proposition 4.4.** *The map  $\theta : \text{Im}\mathcal{R}^+/\mathcal{I}^+ \longrightarrow \text{Im}\mathcal{R}^-/\mathcal{I}^-$  defined by*

$$\theta(\mathcal{R}^+(0_q, X, Z) + \mathcal{I}_q^+) = \mathcal{R}^-(0_q, X, Z) + \mathcal{I}_q^-$$

*is an isomorphism of transitive Lie algebroids.*

*Proof.* We first show that  $\theta$  is well-defined. To do so, suppose  $\mathcal{R}^+(0_q, X, Z) \equiv \mathcal{R}^+(0_q, X', Z') \pmod{\mathcal{I}_q^+}$ . Then there exists  $Z'' \in \mathfrak{h}$  such that

$$\Pi_{\mathfrak{h}}(X - X') + [Z - Z', q] = 0,$$

$$R^-(q)(-\iota(Z - Z') + R^+(q)(X - X')) = \iota Z'',$$

and

$$-(Z - Z') + \Pi_{\mathfrak{h}}(R^+(q)(X - X')) = \text{ad}_q Z''.$$

From these equations, we infer that

$$-\iota(Z - Z') + \Pi_{\mathfrak{h}}(R^-(q)(X - X')) = \text{ad}_q(Z'' - 2(Z - Z')),$$

and

$$R^+(q)(-\iota(Z - Z') + R^-(q)(X - X')) = Z'' - 2(Z - Z').$$

Hence we have  $\mathcal{R}^-(0_q, X, Z) \equiv \mathcal{R}^-(0_q, X', Z') \pmod{\mathcal{I}_q^-}$ , as desired. We shall skip the argument to show that  $\theta$  is 1:1 as it is similar to the one above. The proof of the proposition is therefore complete (it being clear that  $\theta$  is a morphism by Proposition 4.2 (a)).  $\square$

To formulate our next result, introduce the Lie algebroid direct sum  $A\Gamma \oplus_{TU} A\Gamma$ . Clearly, this is the Lie algebroid of the product groupoid  $P = \Gamma \times_{U \times U} \Gamma \rightrightarrows U$  ( $\simeq$  the trivial Lie groupoid  $U \times (G \times G) \times U$ ). For later usage, we shall denote the structure maps (target, source etc.) of  $P$  by  $\alpha_P$ ,  $\beta_P$ , and so forth.

**Theorem 4.5.** (a) *The map  $(\mathcal{R}^+, \mathcal{R}^-) : A^*\Gamma \longrightarrow A\Gamma \oplus_{TU} A\Gamma$  is a monomorphism of transitive Lie algebroids. In particular, the coboundary dynamical Lie algebroid  $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma})$  is integrable.*

(b)  *$\text{Im}(\mathcal{R}^+, \mathcal{R}^-)$  is the Lie subalgebroid*

$$\{(\mathcal{X}_+, \mathcal{X}_-) \in (\text{Im}\mathcal{R}^+ \oplus_{TU} \text{Im}\mathcal{R}^-)_q \mid q \in U, \theta(\mathcal{X}_+ + \mathcal{I}_q^+) = \mathcal{X}_- + \mathcal{I}_q^-\} \quad (4.15)$$

*of  $\text{Im}\mathcal{R}^+ \oplus_{TU} \text{Im}\mathcal{R}^-$ .*

*Proof.* (a) See [L1] for the proof.

(b) Denote by  $A\Gamma_R$  the subbundle of  $Im\mathcal{R}^+ \oplus_{TU} Im\mathcal{R}^-$  defined in (4.15). It is clear that  $Im(\mathcal{R}^+, \mathcal{R}^-) \subset A\Gamma_R$ . Conversely, suppose  $((0_q, X_+, Z), (0_q, X_-, Z)) \in A\Gamma_R$ . Then there exist  $(0_q, X, \tilde{Z}), (0_q, X', \tilde{Z}') \in A\Gamma$  such that  $(0_q, X_+, Z) = \mathcal{R}^+(0_q, X, \tilde{Z})$  and  $(0_q, X_-, Z) = \mathcal{R}^-(0_q, X', \tilde{Z}')$ . Moreover, from the property that  $\theta((0_q, X_+, Z) + \mathcal{I}_q^+) = (0_q, X_-, Z) + \mathcal{I}_q^-$ , we find  $\mathcal{R}^-(0_q, X - X', \tilde{Z} - \tilde{Z}') \equiv 0 \pmod{\mathcal{I}_q^-}$ . Let  $X'' = -\iota(\tilde{Z} - \tilde{Z}') + R^-(q)(X - X')$ . Then it follows from the definition of  $\mathcal{I}_q^-$  that there exists  $Z'' \in \mathfrak{h}$  such that  $\mathcal{R}^+(0_q, X'', Z'') = 0$ . Now, consider the element  $(0_q, X + \frac{1}{2}X'', \tilde{Z} + \frac{1}{2}Z'') \in A\Gamma$ . Clearly,  $\mathcal{R}^+(0_q, X + \frac{1}{2}X'', \tilde{Z} + \frac{1}{2}Z'') = (0_q, X_+, Z)$ . On the other hand,

$$\begin{aligned} & \mathcal{R}^-(0_q, X + \frac{1}{2}X'', \tilde{Z} + \frac{1}{2}Z'') \\ &= (0_q, X_-, Z) + (0_q, X'', 0) + \frac{1}{2}\mathcal{R}^-(0_q, X'', Z''). \end{aligned}$$

But as

$$\begin{aligned} & \mathcal{R}^-(0_q, X'', Z'') \\ &= \mathcal{R}^+(0_q, X'', Z'') - (0_q, 2X'', 0) \\ &= - (0_q, 2X'', 0), \end{aligned}$$

it follows from the above that  $\mathcal{R}^-(0_q, X + \frac{1}{2}X'', \tilde{Z} + \frac{1}{2}Z'') = (0_q, X_-, Z)$ . Thus we have shown that  $((0_q, X_+, Z), (0_q, X_-, Z)) \in Im(\mathcal{R}^+, \mathcal{R}^-)$ .  $\square$

The connection between (mDYBE) and our factorization theory is contained in the decomposition

$$\begin{aligned} (0_q, X, 0) &= \frac{1}{2}\mathcal{R}^+(0_q, X, 0) \\ &\quad - \frac{1}{2}\mathcal{R}^-(0_q, X, 0) \end{aligned} \tag{4.16}$$

where the element  $(0_q, X, 0)$  on the left hand side of (4.16) is in the adjoint bundle  $Ker a$  of  $A\Gamma$ . The reader should note that the vector bundles  $\{\mathcal{R}^\pm(0_q, X, 0) \mid q \in U, X \in \mathfrak{g}\}$  are not Lie subalgebroids of  $A\Gamma$  unless  $R$  is a constant r-matrix. As we pointed out in [L1], this fact has repercussion when we try to formulate a global version of the decomposition in (4.16) (see Corollary 4.6 below).

In the rest of the section, we shall assume both  $G$  and  $U$  are simply-connected. Let  $\Gamma^*$  be the unique source-simply connected Lie groupoid which integrates  $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma})$ . Then  $(\mathcal{R}^+, \mathcal{R}^-)$  can be lifted up to a unique monomorphism of Lie groupoids  $\Gamma^* \longrightarrow \Gamma \times_{U \times U} \Gamma$  which we shall denote by the same symbol. Now, denote by  $\mathcal{I}\Gamma = \{(u, g, u) \mid u \in U, g \in G\}$  the gauge group bundle of  $\Gamma$ . We let  $j : \Gamma \times_{U \times U} \Gamma \longrightarrow \mathcal{I}\Gamma$  be the map defined by  $j(a, b) = ab^{-1}$  and let  $\tilde{m} = j \circ (\mathcal{R}^+, \mathcal{R}^-)$ .

For the sake of completeness, we include the following Corollary of Theorem 4.5 (a) which (essentially) gives a global version of the decomposition in (4.16) which we mentioned above (the reader can find the proof in [L1]). For its formulation, note that the Lie groupoid of  $\{(0_q, 0, Z) \mid q \in U, Z \in \mathfrak{h}\} \subset A^*\Gamma$  is  $H \times U$ , with target and source maps  $\alpha'(h, u) = u$ ,  $\beta'(h, u) = Ad_h u$  and multiplication map  $m'((h, u), (k, Ad_h u)) = (kh, u)$  (this is isomorphic to the Hamiltonian unit in [LP2]). On the other hand, the Lie groupoid of  $\mathcal{R}^\pm \{(0_q, 0, Z) \mid q \in U, Z \in \mathfrak{h}\}$  is given by  $E = \{(u, h, Ad_{h^{-1}} u) \mid u \in U, h \in H\}$  and  $\mathcal{R}^\pm$  embeds  $H \times U$  in  $E$ ,  $\mathcal{R}^\pm \mid H \times U : (h, u) \mapsto (u, h^{-1}, Ad_h u)$ . Clearly, the diagonal  $\Delta(E)$  of  $E \times_{U \times U} E$  acts on  $Im(\mathcal{R}^+, \mathcal{R}^-)$  from the right via the simple formula

$$\begin{aligned} & ((u, k_+, v), (u, k_-, v)) \cdot ((v, h, Ad_{h^{-1}} v), (v, h, Ad_{h^{-1}} v)) \\ &= ((u, k_+ h, Ad_{h^{-1}} v), (u, k_- h, Ad_{h^{-1}} v)) \end{aligned}$$

and the map  $j \mid Im(\mathcal{R}^+, \mathcal{R}^-)$  is constant on the orbits of this action.

**Corollary 4.6.** *Suppose  $U$  is simply-connected, then  $j \mid Im(\mathcal{R}^+, \mathcal{R}^-)$  induces a one-to-one map  $\hat{j} : Im(\mathcal{R}^+, \mathcal{R}^-)/\Delta(E) \longrightarrow \mathcal{I}\Gamma$ . Therefore, for each  $\gamma \in Im \tilde{m}$ , there exists unique  $[(\gamma_+, \gamma_-)]$  in the homogeneous space  $Im(\mathcal{R}^+, \mathcal{R}^-)/\Delta(E)$  such that  $\hat{j}([( \gamma_+, \gamma_-)]) = \gamma$ .*

Let  $f \in I(\mathfrak{g})$  and consider the Hamilton's equation generated by  $F = Pr_3^* f$ . Then according to Corollary 2.7 (a), we can express its restriction to the invariant manifold  $\gamma^{-1}(0) = U \times \{0\} \times \mathfrak{g}$  in the form

$$\begin{aligned} & \frac{d}{dt}(q, 0, X) \\ &= (\Pi_{\mathfrak{h}} df(X), 0, [X, R(q)df(X)]). \end{aligned} \tag{4.17}$$

In the next theorem, we shall express the solution of (4.17) using the adjoint representation of  $\Gamma$  on its adjoint bundle  $Ker a$ , defined by  $\mathbf{Ad}_\gamma(q, 0, X) = (q', 0, Ad_k X)$ , for  $\gamma = (q', k, q) \in \Gamma$ . We shall also make the identifications  $A\Gamma$ ,  $A^*\Gamma \simeq U \times \mathfrak{h} \times \mathfrak{g}$  throughout. Thus the element  $(0, 0, df(X_0))$  which appears in the theorem below will denote the constant section of  $Ker a$  such that  $(0, 0, df(X_0))(q) = (q, 0, df(X_0))$  for  $q \in U$ .

**Theorem 4.7.** *Suppose that  $f \in I(\mathfrak{g})$ ,  $F = Pr_3^* f$  and  $q_0 \in U$ , where  $U$  is simply connected. Then for some  $0 < T \leq \infty$ , there exists a unique element  $(\gamma_+(t), \gamma_-(t)) = ((q_0, k_+(t), q(t)), (q_0, k_-(t), q(t))) \in Im(\mathcal{R}^+, \mathcal{R}^-)$  for  $0 \leq t < T$  which is smooth in  $t$ , solves the factorization problem*

$$\exp\{2t(0, 0, df(X_0))\}(q_0) = \gamma_+(t) \gamma_-(t)^{-1} \quad (4.18)$$

and satisfies

$$(T_{\gamma_+(t)} \mathbf{l}_{\gamma_+(t)^{-1} \dot{\gamma}_+(t)}, T_{\gamma_-(t)} \mathbf{l}_{\gamma_-(t)^{-1} \dot{\gamma}_-(t)}) \in (\mathcal{R}^+, \mathcal{R}^-)(\{q(t)\} \times \{0\} \times \mathfrak{g}) \quad (4.19a)$$

with

$$\gamma_{\pm}(0) = (q_0, 1, q_0). \quad (4.19b)$$

Moreover, the solution of (4.17) with initial data  $(q, 0, X)(0) = (q_0, 0, X_0)$  (i.e. the induced flow on  $\gamma^{-1}(0)$  generated by  $F$ ) is given by the formula

$$(q(t), 0, X(t)) = \mathbf{Ad}_{\gamma_{\pm}(t)^{-1}}(q_0, 0, X_0). \quad (4.20)$$

*Proof.* The uniqueness of the element  $(\gamma_+(t), \gamma_-(t))$  is proved in the same way as in [L1] and makes crucial use of Corollary 4.6.

Assuming the existence of the factors for the moment, we claim that  $(q(t), 0, X(t))$  as given by (4.20) solves (4.17). First of all, we have

$$\begin{aligned} & \mathbf{Ad}_{\gamma_+(t)^{-1}}(q_0, 0, X_0) \\ &= (q(t), 0, Ad_{k_+(t)^{-1}} X_0) \\ &= (q(t), 0, Ad_{k_-(t)^{-1}} Ad_{e^{-2t df(X_0)}} X_0) \\ &= (q(t), 0, Ad_{k_-(t)^{-1}} X_0) \\ &= \mathbf{Ad}_{\gamma_-(t)^{-1}}(q_0, 0, X_0) \end{aligned}$$

where we have used the fact that  $[df(X_0), X_0] = 0$ . Take

$$(q(t), 0, X(t)) = \mathbf{Ad}_{\gamma_+(t)^{-1}}(q_0, 0, X_0).$$

By differentiating the expression, we have

$$\begin{aligned} & \frac{d}{dt}(q(t), 0, X(t)) \\ &= (\dot{q}(t), 0, [X(t), T_{k_+(t)} l_{k_+(t)^{-1}} \dot{k}_+(t)]). \end{aligned} \quad (*)$$

On the other hand, by rewriting (4.18) in the form

$$\exp\{2t(0, df(X_0), 0)\}(q_0) \gamma_-(t) = \gamma_+(t),$$

we have, upon differentiation, that

$$\begin{aligned} & T_{\gamma_+(t)} \mathbf{l}_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t) - T_{\gamma_-(t)} \mathbf{l}_{\gamma_-(t)^{-1}} \dot{\gamma}_-(t) \\ &= 2 \mathbf{Ad}_{\gamma_-(t)^{-1}}(q_0, 0, df(X_0)). \end{aligned}$$

But

$$\begin{aligned} & \mathbf{Ad}_{\gamma_-(t)^{-1}}(q_0, 0, df(X_0)) \\ &= (q(t), 0, Ad_{k_-(t)^{-1}} df(X_0)) \\ &= (q(t), 0, df(X(t))) \end{aligned}$$

as  $f \in I(\mathfrak{g})$ . Hence it follows that

$$\begin{aligned} & T_{\gamma_+(t)} \mathbf{l}_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t) - T_{\gamma_-(t)} \mathbf{l}_{\gamma_-(t)^{-1}} \dot{\gamma}_-(t) \\ &= 2(q(t), 0, df(X(t))). \end{aligned}$$

From the property of  $\gamma_{\pm}$  in (4.19), we can now conclude that

$$T_{\gamma_{\pm}(t)} \mathbf{l}_{\gamma_{\pm}(t)^{-1}} \dot{\gamma}_{\pm}(t) = \mathcal{R}^{\pm}(q(t), 0, df(X(t))).$$

But

$$T_{\gamma_+(t)} \mathbf{l}_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t) = (q(t), \dot{q}(t), T_{k_+(t)} l_{k_+(t)^{-1}} \dot{k}_+(t)),$$

while

$$\mathcal{R}^+(q(t), 0, df(X(t))) = (q(t), \Pi_{\mathfrak{h}} df(X(t)), R^+(q(t)) df(X(t))).$$

By equating the two expressions, we obtain

$$\dot{q}(t) = \Pi_{\mathfrak{h}} df(X(t)),$$

and

$$T_{k_+(t)} l_{k_+(t)^{-1}} \dot{k}_+(t) = R^+(q(t)) df(X(t)).$$

Therefore, on substituting into (\*), we find

$$\begin{aligned} & \frac{d}{dt}(q(t), 0, X(t)) \\ &= (\Pi_{\mathfrak{h}} df(X(t)), [X(t), R(q(t)) df(X(t))]), \end{aligned}$$

as claimed.

To prove the existence of the factors  $\gamma_{\pm}(t)$ , simply solve the initial value problems

$$\dot{k}_{\pm}(t) = T_e l_{k_{\pm}(t)} R^{\pm}(q(t)) df(X(t)), \quad k_{\pm}(0) = 1, \quad (**)$$

where  $q(t)$ ,  $X(t)$  are the solutions of (4.17) with initial data  $(q, 0, X)(0) = (q_0, 0, X_0)$  (which are known to exist by ODE theory). Set  $\gamma_{\pm}(t) = (q_0, k_{\pm}(t), q(t))$ . As can be easily verified, we can combine the equations for  $q(t)$ ,  $k_{\pm}(t)$  into one single equation for  $(\gamma_+(t), \gamma_-(t))$ :

$$\begin{aligned} & \frac{d}{dt}(\gamma_+(t), \gamma_-(t)) \\ &= (T_{\epsilon(q(t))} \mathbf{l}_{\gamma_+(t)} \mathcal{R}^+(q(t), 0, df(X(t))), T_{\epsilon(q(t))} \mathbf{l}_{\gamma_-(t)} \mathcal{R}^-(q(t), 0, df(X(t)))) \\ &= T_{\epsilon_P(\beta_P(\gamma_+(t), \gamma_-(t)))} \mathbf{l}_{(\gamma_+(t), \gamma_-(t))}^P (\mathcal{R}^+, \mathcal{R}^-)(q(t), 0, df(X(t))) \quad (***) \end{aligned}$$

where  $\mathbf{l}_{(\gamma_+(t), \gamma_-(t))}^P$  represents left translation by  $(\gamma_+(t), \gamma_-(t))$  in the product groupoid  $P = \Gamma \times_{U \times U} \Gamma \rightrightarrows U$ . Clearly, what we have just written down is a well-defined equation for  $(\gamma_+(t), \gamma_-(t)) \in \text{Im}(\mathcal{R}^+, \mathcal{R}^-)$ . Moreover, from the initial conditions for  $k_{\pm}(t)$  and  $q(t)$ , we have  $(\gamma_+(0), \gamma_-(0)) \in \text{Im}(\mathcal{R}^+, \mathcal{R}^-)$ .

Now, from the equations for  $k_{\pm}$  in (\*\*), we find

$$\begin{aligned} & T_{\gamma_+(t)\gamma_-(t)^{-1}} \mathbf{l}_{(\gamma_+(t)\gamma_-(t)^{-1})^{-1}} \frac{d}{dt} \gamma_+(t) \gamma_-(t)^{-1} \\ &= (q_0, 0, 2 df(Ad_{k_-(t)} X(t))). \end{aligned}$$

But from the equation for  $k_-(t)$  and  $X(t)$ , we have

$$\begin{aligned} & \frac{d}{dt} Ad_{k_-(t)} X(t) \\ &= Ad_{k_-(t)} \dot{X}(t) + [T_{k_-(t)} r_{k_-(t)^{-1}} \dot{k}_-(t), Ad_{k_-(t)} X(t)] \\ &= [Ad_{k_-(t)} X(t), Ad_{k_-(t)} R(q(t)) df(X(t))] \\ & \quad + [Ad_{k_-(t)} R^-(q(t)) df(X(t)), Ad_{k_-(t)} X(t)] \\ &= 0. \end{aligned}$$

Therefore,  $Ad_{k_-(t)} X(t) = X_0$  and so

$$\begin{aligned} & T_{\gamma_+(t)\gamma_-(t)^{-1}} \mathbf{l}_{(\gamma_+(t)\gamma_-(t)^{-1})^{-1}} \frac{d}{dt} \gamma_+(t) \gamma_-(t)^{-1} \\ &= (q_0, 0, 2 df(X_0)). \end{aligned}$$

As  $\gamma_+(t) \gamma_-(t)^{-1} = (q_0, k_+(t) k_-(t)^{-1}, q_0)$ , this shows that  $k_+(t) k_-(t)^{-1} = e^{2t df(X_0)}$  and consequently,

$$\exp\{2t(0, df(X_0), 0)\}(q_0) = \gamma_+(t) \gamma_-(t)^{-1}.$$

Thus it remains to show that condition (4.19 a) is satisfied. But this is immediate from (\*\*\*). This completes the proof.  $\square$

**Corollary 4.8.** *Let  $\psi_t$  be the induced flow on  $\gamma^{-1}(0) = U \times \{0\} \times \mathfrak{g}$  as defined in (4.20) and let  $\phi_t$  be the Hamiltonian flow of  $\mathcal{F} = L^*f$  on  $X$ , where  $L = Pr_3 \circ \rho$  for a realization map  $\rho : X \longrightarrow A\Gamma$  satisfying A1-A3. If we can solve for  $\phi_t(x)$ ,  $x \in J^{-1}(\mu)$  explicitly from the relation  $\rho(\phi_t)(x) = \psi_t(\rho(x))$ , then the formula  $\phi_t^{red} \circ \pi_\mu = \pi_\mu \circ \phi_t \circ i_\mu$  gives an explicit expression for the flow of the reduced Hamiltonian  $\mathcal{F}_\mu = \widehat{\rho}^* \overline{f}$ .*

**Remark 4.9** (a) The reader should not feel uneasy about the use of the equation (\*\*) above (which involve the solutions  $q(t)$  and  $X(t)$ ) to show the existence of the factors  $k_\pm(t)$ , and which are then used in turn to construct  $q(t)$  and  $X(t)$ . As the reader will see in Section 6 below, knowledge of the existence of the factorization facilitates its construction.

(b) If we take  $K = \frac{1}{2}id_{\mathfrak{g}}$ , which is what we will need in Section 6, then the factorization problem in (4.18) has to be replaced by  $\exp\{t(0, 0, df(X_0))\}(q_0) = \gamma_+(t) \gamma_-(t)^{-1}$ . Otherwise, the solution formula is the same as before.

(c) There is a similar method for solving the Hamiltonian flows generated by natural invariant functions on the gauge group bundles of coboundary dynamical Poisson groupoids. We shall refer the reader to [L1] for details.

(d) For the hyperbolic spin Calogero-Moser systems and the spin Toda lattices which we introduce in the next section, the assumption in Corollary 4.8 (namely, we can solve for  $\phi_t(x)$ ,  $x \in J^{-1}(\mu)$  explicitly from the relation  $\rho(\phi_t)(x) = \psi_t(\rho(x))$ ) are actually not satisfied in general. As the reader will see, some special structure of these equations still enables us to obtain the Hamiltonian flows on  $J^{-1}(\mu)$  from the induced flows on  $\gamma^{-1}(0)$ .

(e) Clearly, Theorem 4.7 also applies in the case when  $R$  is a constant r-matrix. However, it is possible to formulate an analog of this result using the fact that the vector bundles  $\{\mathcal{R}^\pm(0_q, X, 0) \mid q \in U, X \in \mathfrak{g}\}$  are Lie subalgebroids of  $A\Gamma$  in this case, but we provide no details here.

## 5. A family of hyperbolic spin Calogero-Moser systems and the spin Toda lattices.

In [EV], the authors classified solutions of (mDYBE) for pairs  $(\mathfrak{g}, \mathfrak{h})$  of Lie algebras, where  $\mathfrak{g}$  is simple, and  $\mathfrak{h} \subset \mathfrak{g}$  is a Cartan subalgebra. The purpose of this section is to introduce a natural family of hyperbolic spin Calogero-Moser systems associated with these solutions as another application of Proposition 4.2

(a). Remarkably, these models admit scaling limits, and the result is a family of Hamiltonian systems which may be regarded as a spin generalization of the Toda lattice.

Let us begin with some notation. Let  $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$  the root space decomposition of the simple Lie algebra  $\mathfrak{g}$  and let  $(\cdot, \cdot)$  denote its Killing form. For each  $\alpha \in \Delta$ , denote by  $H_\alpha$  the element in  $\mathfrak{h}$  which corresponds to  $\alpha$  under the isomorphism between  $\mathfrak{h}$  and  $\mathfrak{h}^*$  induced by the Killing form  $(\cdot, \cdot)$ . We fix a simple system of roots  $\pi = \{\alpha_1, \dots, \alpha_N\}$  and denote by  $\Delta^\pm$  the corresponding positive/negative system. For any positive root  $\alpha \in \Delta^+$ , we choose root vectors  $e_\alpha \in \mathfrak{g}_\alpha$  and  $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$  which are dual with respect to  $(\cdot, \cdot)$  so that  $[e_\alpha, e_{-\alpha}] = H_\alpha$ . We also fix an orthonormal basis  $(x_i)_{1 \leq i \leq N}$  of  $\mathfrak{h}$ . Lastly, for a subset of simple roots  $\pi' \subset \pi$ , we shall denote the root span of  $\pi'$  by  $\langle \pi' \rangle \subset \Delta$  and set  $\overline{\pi'}^\pm = \Delta^\pm \setminus \langle \pi' \rangle^\pm$ .

For any subset  $\pi' \subset \pi$ , we consider the following  $H$ -equivariant solution of the (mDYBE) (with  $K = \frac{1}{2}id_{\mathfrak{g}}$ ):

$$R(q)X = - \sum_{\alpha \in \Delta} \phi_\alpha(q) X_\alpha e_\alpha \quad (5.1a)$$

where

$$\begin{aligned} \phi_\alpha(q) &= \frac{1}{2} \text{ for } \alpha \in \overline{\pi'}^+, \quad \phi_\alpha(q) = -\frac{1}{2} \text{ for } \alpha \in \overline{\pi'}^- \\ \phi_\alpha(q) &= \frac{1}{2} \coth\left(\frac{1}{2}(\alpha(q))\right) \text{ for } \alpha \in \langle \pi' \rangle, \end{aligned} \quad (5.1b)$$

and  $X_\alpha = (X, e_{-\alpha})$ ,  $\alpha \in \Delta$ . From now onwards, we shall assume  $G$  and  $H$  are simply-connected.

Consider now the coboundary dynamical Lie algebroid  $A^*\Gamma$  which corresponds to this particular choice of  $R$ . By Proposition 4.2 (a), we know that the associated bundle maps  $\mathcal{R}^\pm$  are morphisms of Lie algebroids, hence it follows that the dual maps  $(\mathcal{R}^\pm)^* = -\mathcal{R}^\mp$  are Poisson maps, when the domain and target are equipped with the corresponding Lie-Poisson structures. Note that the Lie-Poisson structure  $\{\cdot, \cdot\}_{A^*\Gamma}$  on the dual bundle  $A^*\Gamma \simeq TU \times \mathfrak{g}$  of the trivial Lie algebroid  $A\Gamma$  is a product structure, as is evident from the expression in Remark 2.5. Hence we have  $H$ -equivariant realizations of  $A^*\Gamma$  (the dual of the trivial Lie algebroid  $A\Gamma$ ) in the dual vector bundle  $A\Gamma$  of the dynamical Lie algebroid  $A^*\Gamma$ .

To summarize, we have the following.

**Proposition 5.1.**  *$(\mathcal{R}^\pm)^*$  are  $H$ -equivariant Poisson maps, where  $H$  acts on  $A^*\Gamma$ ,  $A\Gamma \simeq TU \times \mathfrak{g}$  by acting on the factor  $\mathfrak{g}$  by adjoint action.*

To construct the spin Calogero-Moser system associated to the dynamical  $r$ -matrix  $R$  in (5.1), introduce the quadratic function

$$Q(\xi) = \frac{1}{2}(\xi, \xi), \quad \xi \in \mathfrak{g}. \quad (5.2)$$

We shall take  $\rho = (\mathcal{R}^+)^*$  to be our realization map (the other case with  $(\mathcal{R}^-)^*$  is similar) and let  $L = \text{Pr}_3 \circ \rho$ , as in (3.3). Then the spin Calogero-Moser system associated to  $R$  is the Hamiltonian system on  $A^*\Gamma \simeq TU \times \mathfrak{g}$  generated by the Hamiltonian

$$\mathcal{H}(q, p, \xi) = L^*Q(q, p, \xi) \quad (5.3)$$

Write  $p = \sum_i p_i x_i$ ,  $\xi = \sum_i \xi_i x_i + \sum_{\alpha \in \Delta} \xi_\alpha e_\alpha$ , then we have

**Proposition 5.2.** *The Hamiltonian of the spin Calogero-Moser system associated to the dynamical  $r$ -matrix  $R$  in (5.1) is given by*

$$\begin{aligned} \mathcal{H}(q, p, \xi) = & \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_i \xi_i^2 + \frac{1}{2} \sum_i p_i \xi_i \\ & - \frac{1}{8} \sum_{\alpha \in \langle \pi' \rangle} \frac{\xi_\alpha \xi_{-\alpha}}{\sinh^2 \frac{1}{2} \alpha(q)} \end{aligned} \quad (5.4)$$

and is invariant under the Hamiltonian  $H$ -action on  $A^*\Gamma \simeq TU \times \mathfrak{g}$ :

$$h \cdot (q, p, \xi) = (q, p, \text{Ad}_h \xi) \quad (5.5)$$

with momentum map  $J : TU \times \mathfrak{g} \longrightarrow \mathfrak{h}$  given by

$$J(q, p, \xi) = -\Pi_{\mathfrak{h}} \xi. \quad (5.6)$$

Consider the level set  $J^{-1}(0)$  which is invariant under the flow  $\phi_t$  generated by  $\mathcal{H}$ . Since  $J = \gamma \circ \rho$ , where  $\gamma$  is the momentum map in Proposition 3.1, we clearly have  $\rho(J^{-1}(0)) \subset \gamma^{-1}(0)$ . Hence assumptions A1-A3 are satisfied. Therefore, the family of functions  $L^*I(\mathfrak{g})$  Poisson commute on  $J^{-1}(0)$  and hence descend to Poisson commuting functions on the reduced Poisson variety  $J^{-1}(0)/H$ .

**Remark 5.3** Note that if we consider the realization map  $\rho^- = (\mathcal{R}^-)^* = -\mathcal{R}^+$  instead, then we would have the slightly different Hamiltonian

$$\begin{aligned} \mathcal{H}^-(q, p, \xi) = & \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_i \xi_i^2 - \frac{1}{2} \sum_i p_i \xi_i \\ & - \frac{1}{8} \sum_{\alpha \in \langle \pi' \rangle} \frac{\xi_\alpha \xi_{-\alpha}}{\sinh^2 \frac{1}{2} \alpha(q)} \end{aligned}$$

and the associated Lax operator in this case is given by  $L^-(q, p, \xi) = p - R^+(q)\xi$ .

**Proposition 5.4.** *The Hamiltonian equations of motion generated by  $\mathcal{H}$  on  $A^*\Gamma$  are given by*

$$\begin{aligned} \dot{q} &= p + \frac{1}{2}\Pi_{\mathfrak{h}} \xi, \\ \dot{p} &= -\frac{1}{8} \sum_{\alpha \in <\pi'>} \frac{\coth \frac{1}{2}\alpha(q)}{\sinh^2 \frac{1}{2}\alpha(q)} \xi_{\alpha} \xi_{-\alpha} H_{\alpha}, \\ \dot{\xi} &= \left[ \xi, \frac{1}{4}\Pi_{\mathfrak{h}} \xi + \frac{1}{2}p - \frac{1}{4} \sum_{\alpha \in <\pi'>} \frac{\xi_{\alpha}}{\sinh^2 \frac{1}{2}\alpha(q)} e_{\alpha} \right] \\ &= [\xi, R^+(q)L(q, p, \xi)]. \end{aligned} \tag{5.7}$$

Moreover, under the Hamiltonian flow, we have

$$\begin{aligned} (\Pi_{\mathfrak{h}} \xi)^{\cdot} &= 0 \\ \dot{L}(q, p, \xi) &= [L(q, p, \xi), R(q)L(q, p, \xi)] \\ &\quad - dR(q)(\Pi_{\mathfrak{h}} \xi)L(q, p, \xi). \end{aligned} \tag{5.8}$$

*Proof.* From the expression for the Poisson bracket in Remark 2.5, the equations of motion are given by  $\dot{q} = \delta_2 \mathcal{H}$ ,  $\dot{p} = -\delta_1 \mathcal{H}$  and  $\dot{\xi} = [\xi, \delta \mathcal{H}]$ . Therefore, (5.7) follows by a direct computation. On the other hand, it follows from the definition of  $\rho$  that  $m(q, p, \xi) = q$  and  $\tau(q, p, \xi) = -\Pi_{\mathfrak{h}} \xi$  in the notation introduced in (3.1)-(3.2). Therefore, (5.8) is a consequence of Theorem 3.2 (c).  $\square$

We shall solve Eqn.(5.7) on the level set  $J^{-1}(0)$  (where  $\Pi_{\mathfrak{h}} \xi = 0$ ) in Section 6 below. In order to write down the equations of motion of the reduced Hamiltonian system, we have to restrict to a smooth component of  $J^{-1}(0)/H = U \times \mathfrak{h} \times (\mathfrak{h}^{\perp}/H)$ . For this purpose, we consider the following open submanifold of  $\mathfrak{g}$ :

$$\mathcal{U} = \{ \xi \in \mathfrak{g} \mid \xi_{\alpha_i} = (\xi, e_{-\alpha_i}) \neq 0, \quad i = 1, \dots, N \}. \tag{5.9}$$

Clearly,  $TU \times \mathcal{U}$  is a Poisson submanifold of  $TU \times \mathfrak{g} \simeq A^*\Gamma$  and the  $H$ -action defined by (5.5) induces a Hamiltonian action on  $TU \times \mathcal{U}$ . Therefore, the corresponding momentum map is given by the restriction of the one in (5.6). To simplify notation, we shall denote this momentum map also by  $J$  so that  $J^{-1}(0) = TU \times (\mathfrak{h}^{\perp} \cap \mathcal{U})$ .

Now, recall from [LX2] that the formula

$$g(\xi) = \exp \left( \sum_{i=1}^N \sum_{j=1}^N (C_{ji} \log \xi_{\alpha_j}) h_{\alpha_i} \right) \tag{5.10}$$

defines an  $H$ -equivariant map  $g : \mathcal{U} \longrightarrow H$ , where  $C = (C_{ij})$  is the inverse of the Cartan matrix and  $h_{\alpha_i} = \frac{2}{(\alpha_i, \alpha_i)} H_{\alpha_i}$ ,  $i = 1, \dots, N$ . Using  $g$ , we can identify the

reduced space  $J^{-1}(0)/H = TU \times (\mathfrak{h}^\perp \cap \mathcal{U}/H)$  with  $TU \times \mathfrak{g}_{red}$ , where  $\mathfrak{g}_{red}$  is the affine subspace  $\epsilon + \sum_{\alpha \in \Delta - \pi} \mathbb{C}e_\alpha$ , and  $\epsilon = \sum_{j=1}^N e_{\alpha_j}$ . Indeed, if we write  $\alpha = \sum_{i=1}^N m_\alpha^i \alpha_i$  for each  $\alpha \in \Delta$ , then the identification map is given by

$$(q, p, [\xi]) \mapsto (q, p, Ad_{g(\xi)^{-1}} \xi), \quad (5.11)$$

where explicitly,

$$Ad_{g(\xi)^{-1}} \xi = \epsilon + \sum_{\alpha \in \Delta - \pi} \xi_\alpha \left( \prod_{i=1}^N \xi_{\alpha_j}^{-m_\alpha^j} \right) e_\alpha. \quad (5.12)$$

Thus the natural projection  $\pi_0 : J^{-1}(0) \longrightarrow TU \times \mathfrak{g}_{red}$  is the map

$$(q, p, \xi) \mapsto (q, p, Ad_{g(\xi)^{-1}} \xi). \quad (5.13)$$

We shall write  $s = \sum_{\alpha \in \Delta} s_\alpha e_\alpha$  for  $s \in \mathfrak{g}_{red}$  (note that  $s_{\alpha_j} = 1$  for  $j = 1, \dots, N$ ). By Poisson reduction [MR], the reduced manifold  $TU \times \mathfrak{g}_{red}$  has a unique Poisson structure which is a product structure, where the second factor  $\mathfrak{g}_{red}$  is equipped with the reduction (at 0) of the Lie-Poisson structure on  $\mathcal{U}$  by the  $H$ -action. Now the symplectic leaves of  $\mathfrak{g}_{red}$  are the symplectic reduction of  $\mathcal{O} \cap \mathcal{U}$  at 0, where  $\mathcal{O} \subset \mathfrak{g}$  is an adjoint orbit [MR]. In other words, any symplectic leaf of  $\mathfrak{g}_{red}$  is of the form  $(\mathcal{O} \cap \mathcal{U} \cap \mathfrak{h}^\perp)/H$ , and we shall denote this by  $\mathcal{O}_{red}$ . Consequently, the symplectic leaves of  $TU \times \mathfrak{g}_{red}$  are of the form  $TU \times \mathcal{O}_{red}$ , which is of dimension equal to  $\dim \mathcal{O}$ . Therefore, if  $\mathcal{H}$  is the Hamiltonian of the hyperbolic spin Calogero-Moser system in (5.4), then its reduction  $\mathcal{H}_0$  on  $TU \times \mathfrak{g}_{red}$  is given by

$$\mathcal{H}_0(q, p, s) = \frac{1}{2} \sum_i p_i^2 - \frac{1}{4} \sum_{\alpha \in \langle \pi' \rangle_+} \frac{s_\alpha s_{-\alpha}}{\sinh^2 \frac{1}{2} \alpha(q)}, \quad (5.14)$$

where  $s \in \mathfrak{g}_{red}$ .

**Remark 5.5** (a) In the special case where  $\pi' = \pi$ , the Hamiltonian system generated by  $\mathcal{H}_0$  is isomorphic to the one in Reshetikhin's paper [R].

(b) The family of integrable hyperbolic spin Calogero-systems constructed in this section are different from the ones in [LX2]. Although they look similar, however, their explicit integration requires different tools. To be more precise, the factorization problems for the systems in [LX2] are associated with infinite dimensional Lie groupoids whose vertex groups are loop groups. The solution of such factorization problems requires the use of algebraic geometry (compare Section 6 and [L2]). On the other hand, from the point of view of proving complete integrability, the two distinct families of hyperbolic systems also require totally different considerations. We shall discuss these matters in subsequent publications.

**Proposition 5.6.** *The Hamiltonian equations of motion generated by  $\mathcal{H}_0$  on the reduced Poisson manifold  $TU \times \mathfrak{g}_{red}$  are given by*

$$\begin{aligned}\dot{q} &= p, \\ \dot{p} &= -\frac{1}{8} \sum_{\alpha \in \langle \pi' \rangle} \frac{\coth \frac{1}{2}\alpha(q)}{\sinh^2 \frac{1}{2}\alpha(q)} s_\alpha s_{-\alpha} H_\alpha, \\ \dot{s} &= [s, \mathcal{M}]\end{aligned}$$

where

$$\mathcal{M} = -\frac{1}{4} \sum_{\alpha \in \langle \pi' \rangle} \frac{s_\alpha}{\sinh^2 \frac{1}{2}\alpha(q)} e_\alpha + \frac{1}{4} \sum_{i,j} C_{ji} \sum_{\substack{\alpha \in \langle \pi' \rangle - \pi' \\ \alpha_j - \alpha \in \Delta}} N_{\alpha, \alpha_j - \alpha} \frac{s_\alpha s_{\alpha_j - \alpha}}{\sinh^2 \frac{1}{2}\alpha(q)} h_{\alpha_i}.$$

(Here we use the notation  $[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha+\beta}$  if  $\alpha + \beta \in \Delta$ .)

*Proof.* The first two equations are obvious from Proposition 5.4 and the definition of  $s$ . To derive the equation of  $s$ , we differentiate  $s = Ad_{g(\xi)}^{-1} \xi$  with respect to  $t$ , assuming  $\xi$  satisfies the equation in Proposition 5.4 with  $\Pi_{\mathfrak{h}} \xi = 0$ . Then we have

$$\begin{aligned}\dot{s} &= \left[ T_{g(\xi)^{-1}} r_{g(\xi)} \frac{d}{dt} g(\xi)^{-1}, s \right] + Ad_{g(\xi)^{-1}} \dot{\xi} \\ &= \left[ s, \frac{1}{2}p - \frac{1}{4} \sum_{\alpha \in \langle \pi' \rangle} \frac{\xi_\alpha}{\sinh^2 \frac{1}{2}\alpha(q)} e^{-\alpha(\log g(\xi))} e_\alpha - T_{g(\xi)^{-1}} r_{g(\xi)} \frac{d}{dt} g(\xi)^{-1} \right]. \quad (*)\end{aligned}$$

By a direct computation, we find  $\xi_\alpha e^{-\alpha(\log g(\xi))} = \xi_\alpha (\prod_{i=1}^N \xi_{\alpha_j}^{-m_j^\alpha}) = s_\alpha$ . Meanwhile, by differentiating  $g(\xi)^{-1}$ , we obtain

$$-T_{g(\xi)^{-1}} r_{g(\xi)} \frac{d}{dt} g(\xi)^{-1} = \sum_{i,j} C_{ji} \dot{\xi}_{\alpha_j} \xi_{\alpha_j}^{-1} h_{\alpha_i}.$$

But

$$\begin{aligned}\dot{\xi}_{\alpha_j} &= (\dot{\xi}, e_{-\alpha_j}) \\ &= \left( \left[ \xi, \frac{1}{2}p - \frac{1}{4} \sum_{\alpha \in \langle \pi' \rangle} \frac{\xi_\alpha}{\sinh^2 \frac{1}{2}\alpha(q)} e_\alpha \right], e_{-\alpha_j} \right) \\ &= -\frac{1}{2} \alpha_j(p) \xi_{\alpha_j} + \frac{1}{4} \xi_{\alpha_j} \sum_{\substack{\alpha \in \langle \pi' \rangle - \pi' \\ \alpha_j - \alpha \in \Delta}} N_{\alpha, \alpha_j - \alpha} \frac{s_\alpha s_{\alpha_j - \alpha}}{\sinh^2 \frac{1}{2}\alpha(q)}\end{aligned}$$

whereas  $\frac{1}{2}p = \sum_{i,j} C_{ji} \alpha_j(\frac{1}{2}p) h_{\alpha_i}$ . Therefore, on substituting the above expressions into (\*), the desired equation follows.  $\square$

In the rest of the section, we shall describe a scaling limit of the hyperbolic spin Calogero-Moser systems. More precisely, we consider

$$\begin{aligned} q &= x + 2\tau w, \quad \tau > 0, \\ \xi_i &= \eta_i, \quad 1 \leq i \leq N, \\ \xi_\alpha &= \eta_\alpha e^\tau, \quad \alpha \in \Delta, \end{aligned} \tag{5.15}$$

in the limit  $\tau \rightarrow \infty$ , where

$$w = \sum_{\alpha \in \Delta^+} \frac{H_\alpha}{(\alpha, \alpha)}. \tag{5.16}$$

Note that this is analogous to the one in [DP], where the standard (spinless) elliptic Calogero-Moser system was considered. Clearly,  $\alpha(w) = (\alpha, \delta^\vee)$ , where

$$\delta^\vee = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta^\vee, \quad \beta^\vee = \frac{2\beta}{(\beta, \beta)}. \tag{5.17}$$

If for  $\alpha \in \Delta$ , we write  $\alpha = \sum_{i=1}^N m_\alpha^i \alpha_i$ , then it is not hard to show that

$$l(\alpha) := \alpha(w) = \sum_{i=1}^N m_\alpha^i. \tag{5.18}$$

Therefore,  $l(\alpha)$  is the level (or height) of  $\alpha$ . Hence  $l(\alpha)$  is an integer, and assumes the value 1 if and only if  $\alpha \in \pi$ .

Now, with the definition of  $x$  and  $\eta_\alpha$  in (5.15), it is easy to show that for  $\alpha \in <\pi' >^+$ , we have

$$\frac{\xi_\alpha \xi_{-\alpha}}{\sinh^2 \frac{1}{2} \alpha(q)} \sim 4\eta_\alpha \eta_{-\alpha} e^{-\alpha(x) - 2\tau(l(\alpha)-1)}, \quad \tau \rightarrow \infty. \tag{5.19}$$

Therefore, if  $\alpha \in <\pi' > -\pi'$ , we have

$$\lim_{\tau \rightarrow \infty} \frac{\xi_\alpha \xi_{-\alpha}}{\sinh^2 \frac{1}{2} \alpha(q)} = 0. \tag{5.20}$$

On the other hand, if  $\alpha \in \pi'$ , we obtain

$$\lim_{\tau \rightarrow \infty} \frac{\xi_\alpha \xi_{-\alpha}}{\sinh^2 \frac{1}{2} \alpha(q)} = 4\eta_\alpha \eta_{-\alpha} e^{-\alpha(x)}. \tag{5.21}$$

Accordingly, the scaling limit of the Hamiltonian  $\mathcal{H}$  of the hyperbolic spin Calogero-Moser system is given by

$$\begin{aligned} \mathcal{H}^s(x, p, \eta) &= \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_i \eta_i^2 + \frac{1}{2} \sum_i p_i \eta_i \\ &\quad - \sum_{\alpha \in \pi'} \eta_\alpha \eta_{-\alpha} e^{-\alpha(x)}. \end{aligned} \tag{5.22}$$

Note that in contrast to the spinless case, we do not know a priori the Poisson manifold on which  $\mathcal{H}^s$  is defined. This issue will be settled below, but first we shall work out the scaling limits of the Hamiltonian equations of motion and the (quasi) Lax equation in Proposition 5.4 which will in fact give us some clue to this problem.

Let

$$\eta = \sum_{i=1}^N \eta_i x_i + \sum_{\alpha \in \Delta} \eta_\alpha e_\alpha. \quad (5.23)$$

**Proposition 5.7.** *The scaling limit of the Hamiltonian equations of motion in (5.7) is given by*

$$\begin{aligned} \dot{x} &= p + \frac{1}{2} \Pi_{\mathfrak{h}} \eta, \\ \dot{p} &= - \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_\alpha \eta_{-\alpha} H_\alpha, \\ \dot{\eta} &= \left[ \eta, \frac{1}{4} \Pi_{\mathfrak{h}} \eta + \frac{1}{2} p \right]. \end{aligned} \quad (5.24)$$

*Proof.* The equation for  $x$  is obvious from the equation for  $q$ . On the other hand, the equation for  $p$  is a consequence of our previous analysis in (5.20)-(5.21) and the fact that  $\coth \frac{1}{2} \alpha(q) \rightarrow 1$  as  $\tau \rightarrow \infty$  for  $\alpha \in \Delta^+$ . To get the last equation above, we make the substitution from (5.15) into the equation for  $\xi$  and then divide both sides by  $e^\tau$ , this gives

$$\begin{aligned} & (\Pi_{\mathfrak{h}^\perp} \eta)^\cdot \\ &= \left[ e^{-\tau} \Pi_{\mathfrak{h}} \eta + \Pi_{\mathfrak{h}^\perp} \eta, \frac{1}{4} \Pi_{\mathfrak{h}} \eta + \frac{1}{2} p - \frac{1}{4} \sum_{\alpha \in \langle \pi' \rangle} \frac{\eta_\alpha e^\tau}{\sinh^2 \frac{1}{2} \alpha(x + 2\tau w)} e_\alpha \right] \end{aligned}$$

as  $(\Pi_{\mathfrak{h}} \eta)^\cdot = 0$ . Therefore, upon letting  $\tau \rightarrow \infty$ , we find

$$(\Pi_{\mathfrak{h}^\perp} \eta)^\cdot = \left[ \Pi_{\mathfrak{h}^\perp} \eta, \frac{1}{4} \Pi_{\mathfrak{h}} \eta + \frac{1}{2} p \right].$$

Combining this with  $(\Pi_{\mathfrak{h}} \eta)^\cdot = 0$ , the equation for  $\eta$  follows.  $\square$

At this juncture, we remark that the Lax operator  $L(q, p, \xi)$  does not actually admit a finite limit, as can be easily verified. However, we can remedy this by considering the following gauge-equivalent equation:

$$\begin{aligned} (Ad_{e^{-\tau w}} L)^\cdot &= [Ad_{e^{-\tau w}} L, Ad_{e^{-\tau w}} R(q) L] \\ &\quad - Ad_{e^{-\tau w}} dR(q) (\Pi_{\mathfrak{h}} \xi) L. \end{aligned} \quad (5.25)$$

Thus we introduce

$$\begin{aligned} L_\tau(x, p, \eta) &:= Ad_{e^{-\tau w}} L(x + 2\tau w, p, \Pi_{\mathfrak{h}} \eta + e^\tau \Pi_{\mathfrak{h}^\perp} \eta), \\ M_\tau(x, p, \eta) &:= Ad_{e^{-\tau w}} R(x + 2\tau w) L(x + 2\tau w, p, \Pi_{\mathfrak{h}} \eta + e^\tau \Pi_{\mathfrak{h}^\perp} \eta). \end{aligned} \quad (5.26)$$

Using the relation  $Ad_{e^{-\tau w}} e_\alpha = e^{-\tau l(\alpha)} e_\alpha$  and the  $H$ -equivariance of  $R$ , we easily find that

$$\begin{aligned} L_\tau(x, p, \eta) &= p + \frac{1}{2} \Pi_{\mathfrak{h}} \eta + \sum_{\alpha \in \langle \pi' \rangle} \frac{e^{\frac{1}{2}\alpha(x+2\tau w)}}{2 \sinh \frac{1}{2}\alpha(x+2\tau w)} \eta_\alpha e^{-\tau(l(\alpha)-1)} e_\alpha \\ &\quad + \sum_{\alpha \in \overline{\pi'}^+} \eta_\alpha e^{-\tau(l(\alpha)-1)} e_\alpha, \end{aligned} \quad (5.27)$$

whereas

$$\begin{aligned} M_\tau(x, p, \eta) &= -\frac{1}{2} \sum_{\alpha \in \langle \pi' \rangle} \coth \frac{1}{2}\alpha(x+2\tau w) \frac{e^{\frac{1}{2}\alpha(x+2\tau w)}}{2 \sinh \frac{1}{2}\alpha(x+2\tau w)} \eta_\alpha e^{-\tau(l(\alpha)-1)} e_\alpha \\ &\quad + \frac{1}{2} \sum_{\alpha \in \overline{\pi'}^+} \eta_\alpha e^{-\tau(l(\alpha)-1)} e_\alpha. \end{aligned} \quad (5.28)$$

Now for  $\alpha \in \langle \pi' \rangle^+$ , we have

$$\frac{e^{\frac{1}{2}\alpha(x+2\tau w)}}{2 \sinh \frac{1}{2}\alpha(x+2\tau w)} e^{-\tau(l(\alpha)-1)} \sim e^{-\tau(l(\alpha)-1)}, \quad \tau \rightarrow \infty. \quad (5.29)$$

Similarly, for  $\alpha \in \langle \pi' \rangle^-$ , we obtain

$$\frac{e^{\frac{1}{2}\alpha(x+2\tau w)}}{2 \sinh \frac{1}{2}\alpha(x+2\tau w)} e^{-\tau(l(\alpha)-1)} \sim -e^{\alpha(x)+\tau(l(\alpha)+1)}, \quad \tau \rightarrow \infty. \quad (5.30)$$

**Proposition 5.8.** *We have*

$$\begin{aligned} \mathbf{L}(x, p, \eta) &:= \lim_{\tau \rightarrow \infty} L_\tau(x, p, \eta) \\ &= p + \frac{1}{2} \Pi_{\mathfrak{h}} \eta + \sum_{\alpha \in \pi} \eta_\alpha e_\alpha - \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_{-\alpha} e_{-\alpha}, \end{aligned} \quad (5.31)$$

$$\begin{aligned} \mathbf{M}(x, p, \eta) &:= \lim_{\tau \rightarrow \infty} M_\tau(x, p, \eta) \\ &= -\frac{1}{2} \sum_{\alpha \in \pi} \eta_\alpha e_\alpha - \frac{1}{2} \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_{-\alpha} e_{-\alpha}. \end{aligned} \quad (5.32)$$

Moreover, the scaling limit of the (quasi) Lax equation (5.8) is given by

$$\dot{\mathbf{L}} = [\mathbf{L}, \mathbf{M}] = [\mathbf{L}, \mathbf{R}(\mathbf{L})] \quad (5.33)$$

where  $\mathbf{R}$  is the constant  $r$ -matrix defined by

$$\mathbf{R}(\eta) = \frac{1}{2} \sum_{\alpha \in \Delta^-} \eta_\alpha e_\alpha - \frac{1}{2} \sum_{\alpha \in \Delta^+} \eta_\alpha e_\alpha. \quad (5.34)$$

*Proof.* Using the asymptotics in (5.29)-(5.30), we obtain

$$\lim_{\tau \rightarrow \infty} \frac{e^{\frac{1}{2}\alpha(x+2\tau w)}}{2 \sinh \frac{1}{2}\alpha(x+2\tau w)} e^{-\tau(l(\alpha)-1)} = \begin{cases} 0, & \alpha \in \langle \pi' \rangle - (\pi' \cup (-\pi')) \\ 1, & \alpha \in \pi' \\ -e^{\alpha(x)}, & \alpha \in -\pi' \end{cases}$$

from which the formulas for  $\mathbf{L}$  and  $\mathbf{M}$  follow. Therefore, in order to demonstrate the validity of (5.33), it remains to show that

$$\lim_{\tau \rightarrow \infty} Ad_{e^{-\tau w}} dR(x+2\tau w)(\Pi_{\mathfrak{h}} \eta) L(x+2\tau w, p, \Pi_{\mathfrak{h}} \eta + e^\tau \Pi_{\mathfrak{h}^\perp} \eta) = 0.$$

By the  $H$ -equivariance of  $R$  and its explicit expression,

$$\begin{aligned} & Ad_{e^{-\tau w}} dR(x+2\tau w)(\Pi_{\mathfrak{h}} \eta) L(x+2\tau w, p, \Pi_{\mathfrak{h}} \eta + e^\tau \Pi_{\mathfrak{h}^\perp} \eta) \\ &= dR(x+2\tau w)(\Pi_{\mathfrak{h}} \eta) L_\tau(x, p, \eta) \\ &= \sum_{\alpha \in \langle \pi' \rangle} \alpha(\Pi_{\mathfrak{h}} \eta) \eta_\alpha \frac{e^{\frac{1}{2}\alpha(x+2\tau w)}}{(2 \sinh \frac{1}{2}\alpha(x+2\tau w))^3} e^{-\tau(l(\alpha)-1)} e_\alpha. \end{aligned}$$

But as  $\tau \rightarrow \infty$ , we have

$$\begin{aligned} & \frac{e^{\frac{1}{2}\alpha(x+2\tau w)}}{(2 \sinh \frac{1}{2}\alpha(x+2\tau w))^3} e^{-\tau(l(\alpha)-1)} \\ & \sim \begin{cases} e^{-\alpha(x)-\tau(3l(\alpha)-1)}, & \alpha \in \langle \pi' \rangle^+ \\ -e^{2\alpha(x)+\tau(3l(\alpha)+1)}, & \alpha \in \langle \pi' \rangle^- \end{cases}. \end{aligned}$$

Hence the required property follows.  $\square$

**Remark 5.9** (a) The constant  $r$ -matrix  $\mathbf{R}$  is the scaling limit of the dynamical  $r$ -matrix in the sense that  $\mathbf{R}(\xi) = \lim_{\tau \rightarrow \infty} R(x+2\tau w)\xi$ .

(b) It is a remarkable fact that the (quasi) Lax equation (5.8) scales to the genuine Lax equation in (5.33). In other words, the obstruction to integrability dissolves in the scaling limit.

(c) The reader should note that the scaling limit above is a singular limit. For this reason, the geometric structures are not preserved. As the reader will see

in what follows,  $\mathcal{H}^s$  is defined on a Poisson manifold different from that of  $\mathcal{H}$ . Therefore, it is not surprising that their Hamiltonian realization would require separate consideration.

We now describe a Hamiltonian formulation of the equations in Proposition 5.7. To do so, we consider the trivial Lie algebroid  $S = T\mathfrak{h} \times \mathfrak{g}$  over  $\mathfrak{h}$ , where  $\mathfrak{g}$  is identified with the semi-direct product  $\mathfrak{h} \ltimes \mathfrak{h}^\perp$  associated with the representation  $ad$  of the Cartan subalgebra  $\mathfrak{h}$  in  $\mathfrak{h}^\perp$ . Thus the Lie algebroid bracket on  $S$  is given by

$$\begin{aligned} & [(Z, X), (Z', X')]_S(x) \\ &= (dZ'(x)Z(x) - dZ(x)Z'(x), dX'(x)Z(x) - dX(x)Z'(x)) \\ & \quad + [\Pi_{\mathfrak{h}}X(x), \Pi_{\mathfrak{h}^\perp}X'(x)] - [\Pi_{\mathfrak{h}}X'(x), \Pi_{\mathfrak{h}^\perp}X(x)] \end{aligned} \quad (5.35)$$

where  $Z, Z' : \mathfrak{h} \longrightarrow \mathfrak{h}$ ,  $X, X' : \mathfrak{h} \longrightarrow \mathfrak{g}$  are holomorphic maps and  $x \in \mathfrak{h}$ .

**Proposition 5.10.** *The Lie-Poisson structure on the dual bundle  $S^* \simeq \mathfrak{h} \times \mathfrak{h} \times \mathfrak{g}$  of the trivial Lie algebroid  $S$  is given by*

$$\begin{aligned} & \{ \varphi, \psi \}_{S^*}(x, p, \eta) \\ &= (\delta_1 \psi, \delta_2 \varphi) - (\delta_1 \varphi, \delta_2 \psi) + (\eta, [\Pi_{\mathfrak{h}} \delta \varphi, \Pi_{\mathfrak{h}^\perp} \delta \psi]) + [\Pi_{\mathfrak{h}^\perp} \delta \varphi, \Pi_{\mathfrak{h}} \delta \psi] \end{aligned}$$

and the Hamiltonian equations generated by  $\varphi : S^* \longrightarrow \mathbb{C}$  are:

$$\begin{aligned} \dot{x} &= \delta_2 \varphi, \\ \dot{p} &= -\delta_1 \varphi, \\ \dot{\eta} &= [\eta, \Pi_{\mathfrak{h}} \delta \varphi] + \Pi_{\mathfrak{h}} [\eta, \delta \varphi]. \end{aligned} \quad (5.36)$$

*Proof.* Using the method of calculation in Section 2, we have

$$\begin{aligned} & \{ \varphi, \psi \}_{S^*}(x, p, \eta) \\ &= l_{[s(\varphi), s(\psi)]_S}(x, p, \eta) + (\delta_1 \psi, \delta_2 \varphi) - (\delta_1 \varphi, \delta_2 \psi). \end{aligned}$$

Now, from the expression for  $[\cdot, \cdot]_S$ , it is easy to check that

$$\begin{aligned} & [s(\varphi), s(\psi)]_S(x) \\ &= (0, [\Pi_{\mathfrak{h}} \delta \varphi, \Pi_{\mathfrak{h}^\perp} \delta \psi] - [\Pi_{\mathfrak{h}} \delta \psi, \Pi_{\mathfrak{h}^\perp} \delta \varphi]). \end{aligned}$$

Hence we have

$$\begin{aligned} & l_{[s(\varphi), s(\psi)]_S}(x, p, \eta) \\ &= (\eta, [\Pi_{\mathfrak{h}} \delta \varphi, \Pi_{\mathfrak{h}^\perp} \delta \psi] + [\Pi_{\mathfrak{h}^\perp} \delta \varphi, \Pi_{\mathfrak{h}} \delta \psi]). \end{aligned}$$

Assembling the calculations, we obtain the formula for  $\{ \varphi, \psi \}_{S^*}(x, p, \eta)$ . □

To prepare for our next result, we need to introduce further constructs. First of all, let  $\mathbf{A}^* \simeq T\mathfrak{h} \times \mathfrak{g}$  be the coboundary dynamical Lie algebroid associated with the constant r-matrix  $\mathbf{R}$ . Since  $\mathfrak{h}$  is Abelian, the Lie-Poisson structure on its dual bundle  $\mathbf{A} \simeq T\mathfrak{h} \times \mathfrak{g}$  takes the form

$$\begin{aligned} & \{\varphi, \psi\}_{\mathbf{A}}(x, p, \eta) \\ &= (\eta, [\mathbf{R}(\delta\varphi) - \delta_2\varphi, \delta\psi] + [\delta\varphi, \mathbf{R}(\delta\psi) - \delta_2\psi]) \\ &+ (\delta_1\psi, \Pi_{\mathfrak{h}}\delta\varphi) - (\delta_1\varphi, \Pi_{\mathfrak{h}}\delta\psi). \end{aligned} \quad (5.37)$$

Now, define

$$\begin{aligned} \rho : T\mathfrak{h} \times \mathfrak{g} \simeq S^* &\longrightarrow \mathbf{A} \simeq T\mathfrak{h} \times \mathfrak{g} \\ (x, p, \eta) &\mapsto (x, -\Pi_{\mathfrak{h}}\eta, \mathbf{L}(x, p, \eta)). \end{aligned} \quad (5.38)$$

**Theorem 5.11.**  $\rho$  is an  $H$ -equivariant Poisson map, where the  $H$ -action on  $S^*$  given by

$$\begin{aligned} h \cdot (x, p, \eta) &= (x, p, Ad_h\eta) \\ &= (x, p, \Pi_{\mathfrak{h}}\eta + \Pi_{\mathfrak{h}^\perp}Ad_h\eta) \end{aligned}$$

is Hamiltonian with equivariant momentum map  $\mathbf{J} : T\mathfrak{h} \times \mathfrak{g} \longrightarrow \mathfrak{h}$ ,  $(x, p, \eta) \mapsto -\Pi_{\mathfrak{h}}\eta$ . Moreover, the equations in Proposition 5.7 are the Hamiltonian equations generated by  $\mathcal{H}^s(x, p, \eta) = \mathbf{L}^*Q(x, p, \eta)$  in the Lie-Poisson structure  $\{\cdot, \cdot\}_{S^*}$  and admit  $\mathbf{L}^*I(\mathfrak{g})$  as a family of conserved quantities in involution.

*Proof.* Let  $\varphi, \psi \in C^\infty(\mathbf{A})$ . By direct calculation, we have

$$\begin{aligned} & \delta(\varphi \circ \rho)(x, p, \eta) \\ &= -\delta_2\varphi(\rho(x, p, \eta)) + \frac{1}{2}\Pi_{\mathfrak{h}}\delta\varphi(\rho(x, p, \eta)) \\ &+ \sum_{\alpha \in \pi} (\delta\varphi(\rho(x, p, \eta)))_{-\alpha} e_{-\alpha} - \sum_{\alpha \in \pi'} e^{-\alpha(x)} (\delta\varphi(\rho(x, p, \eta)))_{\alpha} e_{\alpha}, \end{aligned}$$

$\delta_1(\varphi \circ \rho)(x, p, \eta) = \delta_1\varphi(\rho(x, p, \eta)) + \sum_{\alpha \in \pi'} \delta\varphi_{\alpha} \eta_{\alpha} e^{-\alpha(x)} H_{\alpha}$ , and  $\delta_2(\varphi \circ \rho)(x, p, \eta) = \Pi_{\mathfrak{h}}\delta\varphi(\rho(x, p, \eta))$ . To simplify notation, let  $X = \delta\varphi(\rho(x, p, \eta))$ ,  $Y = \delta_1\varphi(\rho(x, p, \eta))$  and  $Z = \delta_2\varphi(\rho(x, p, \eta))$  and denote the corresponding quantities associated with  $\psi$  by  $X'$ ,  $Y'$  and  $Z'$  respectively. Then it follows from the expression of  $\{\cdot, \cdot\}_{S^*}$  in Proposition 5.10 and the above calculation that

$$\begin{aligned} & \{\varphi \circ \rho, \psi \circ \rho\}_{S^*}(x, p, \eta) \\ &= \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_{-\alpha} X'_{\alpha} \alpha(Z + \frac{1}{2}\Pi_{\mathfrak{h}}X) + \sum_{\alpha \in \pi} \eta_{\alpha} X'_{-\alpha} \alpha(Z - \frac{1}{2}\Pi_{\mathfrak{h}}X) \\ &+ (Y', \Pi_{\mathfrak{h}}X) - (X \leftrightarrow X', Y \leftrightarrow Y', Z \leftrightarrow Z'). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \{\varphi, \psi\}_{\mathbf{A}} \circ \rho(x, p, \eta) \\ &= (\mathbf{L}(x, p, \eta), [\mathbf{R}(X) - Z, X'] + [X, \mathbf{R}(X') - Z']) \\ & \quad + (Y', \Pi_{\mathfrak{h}} X) - (Y, \Pi_{\mathfrak{h}} X'). \end{aligned}$$

But

$$\begin{aligned} & \left( \sum_{\alpha \in \pi} \eta_{\alpha} e_{\alpha}, [\mathbf{R}(X) - Z, X'] + [X, \mathbf{R}(X') - Z'] \right) \\ &= \sum_{\alpha \in \pi} \eta_{\alpha} X'_{-\alpha} \alpha \left( Z - \frac{1}{2} \Pi_{\mathfrak{h}} X \right) - (X \leftrightarrow X', Y \leftrightarrow Y', Z \leftrightarrow Z'), \end{aligned}$$

while

$$\begin{aligned} & \left( - \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_{\alpha} e_{\alpha}, [\mathbf{R}(X) - Z, X'] + [X, \mathbf{R}(X') - Z'] \right) \\ &= \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_{-\alpha} X'_{\alpha} \alpha \left( Z + \frac{1}{2} \Pi_{\mathfrak{h}} X \right) - (X \leftrightarrow X', Y \leftrightarrow Y', Z \leftrightarrow Z'). \end{aligned}$$

Putting the calculations together, we conclude that  $\{\varphi, \psi\}_{\mathbf{A}} \circ \rho(x, p, \eta)$  is identical to  $\{\varphi \circ \rho, \psi \circ \rho\}_{S^*}(x, p, \eta)$ . Thus  $\rho$  is a Poisson map. Alternatively, we can also establish the assertion by showing that the dual of the bundle map  $\boldsymbol{\rho}$  is a morphism of Lie algebroids.

To show that the equations in Proposition 5.7 are the Hamiltonian equations generated by  $\mathcal{H}^s$  in the Poisson structure  $\{\cdot, \cdot\}_{S^*}$ , note that

$$\delta_1 \mathcal{H}^s = \sum_{\alpha \in \pi'} \eta_{\alpha} \eta_{-\alpha} e^{-\alpha(x)} H_{\alpha}, \quad \delta_2 \mathcal{H}^s = p + \frac{1}{2} \Pi_{\mathfrak{h}} \eta$$

and

$$\delta \mathcal{H}^s = \frac{1}{2} p + \frac{1}{4} \Pi_{\mathfrak{h}} \eta - \sum_{\alpha \in \pi'} e^{-\alpha(x)} (\eta_{\alpha} e_{\alpha} + \eta_{-\alpha} e_{-\alpha}).$$

From these formulas, it is clear that the equations for  $x$  and  $p$  from (5.36) with  $\varphi = \mathcal{H}^s$  are identical to the corresponding ones in Proposition 5.7. To show that the equations for  $\eta$  are identical as well, it is enough to check that  $[\eta, \delta \mathcal{H}^s] \in \mathfrak{h}^{\perp}$ . Write  $\eta = \Pi_{\mathfrak{h}} \eta + \Pi_{\mathfrak{h}^{\perp}} \eta$  and substitute into  $[\eta, \delta \mathcal{H}^s]$ . As  $[\mathfrak{h}, \mathfrak{h}^{\perp}] \subset \mathfrak{h}^{\perp}$ , we only have to consider the term  $[\Pi_{\mathfrak{h}^{\perp}} \eta, - \sum_{\alpha \in \pi'} e^{-\alpha(x)} (\eta_{\alpha} e_{\alpha} + \eta_{-\alpha} e_{-\alpha})]$  in  $[\eta, \delta \mathcal{H}^s]$ . Expanding out, we have

$$\begin{aligned} & [\Pi_{\mathfrak{h}^{\perp}} \eta, - \sum_{\alpha \in \pi'} e^{-\alpha(x)} (\eta_{\alpha} e_{\alpha} + \eta_{-\alpha} e_{-\alpha})] \\ &= - \sum_{\alpha \in \pi'} \sum_{\beta \in \Delta^+} e^{-\alpha(x)} \eta_{\alpha} \eta_{\beta} [e_{\beta}, e_{\alpha}] - \sum_{\alpha \in \pi'} \sum_{\beta \in \Delta^+} e^{-\alpha(x)} \eta_{-\alpha} \eta_{-\beta} [e_{-\beta}, e_{-\alpha}] \\ & \quad - \sum_{\alpha \in \pi'} \sum_{\beta \in \Delta^+} e^{-\alpha(x)} \eta_{-\alpha} \eta_{\beta} [e_{\beta}, e_{-\alpha}] - \sum_{\alpha \in \pi'} \sum_{\beta \in \Delta^+} e^{-\alpha(x)} \eta_{\alpha} \eta_{-\beta} [e_{-\beta}, e_{\alpha}]. \end{aligned}$$

Clearly, the first two terms of the above sum are in  $\mathfrak{h}^\perp$ . Now,

$$\begin{aligned} & \Pi_{\mathfrak{h}} \left( \sum_{\alpha \in \pi'} \sum_{\beta \in \Delta^+} e^{-\alpha(x)} \eta_{-\alpha} \eta_{\beta} [e_{\beta}, e_{-\alpha}] \right) \\ &= \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_{\alpha} \eta_{-\alpha} H_{\alpha} \end{aligned}$$

while

$$\begin{aligned} & \Pi_{\mathfrak{h}} \left( \sum_{\alpha \in \pi'} \sum_{\beta \in \Delta^+} e^{-\alpha(x)} \eta_{\alpha} \eta_{-\beta} [e_{-\beta}, e_{\alpha}] \right) \\ &= - \sum_{\alpha \in \pi'} e^{-\alpha(x)} \eta_{\alpha} \eta_{-\alpha} H_{\alpha}. \end{aligned}$$

So the sum of the last two terms in the above sum is in  $\mathfrak{h}^\perp$  as well. We shall leave the rest of the proof to the reader.  $\square$

We shall call the Hamiltonian systems generated by  $\mathcal{H}^s$  in the Lie-Poisson structure  $\{\cdot, \cdot\}_{S^*}$  *spin Toda lattices*. To close this section, we shall consider reduction of the spin Toda lattices. As before, we consider the submanifold  $\mathcal{U}$  defined in (5.9). Then clearly, the  $H$ -action defined in Theorem 5.11 induces a Hamiltonian action on  $T\mathfrak{h} \times \mathcal{U}$ . Denote the corresponding momentum map also by  $\mathbf{J}$ , we have  $\mathbf{J}^{-1}(0) = T\mathfrak{h} \times (\mathfrak{h}^\perp \cap \mathcal{U})$ . In this case, it is easy to verify that a generic orbit  $\mathcal{O} \subset \mathfrak{g}$  (recall that  $\mathfrak{g} \simeq \mathfrak{h} \ltimes \mathfrak{h}^\perp$ ) is of dimension  $2N$ , where  $N = \text{rank}(\mathfrak{g})$ . Therefore,  $\mathcal{O}_{red} = (\mathcal{O} \cap \mathcal{U} \cap \mathfrak{h}^\perp)/H$  is a point. Indeed, we have

**Corollary 5.13.** *The reduction of the Hamiltonian  $\mathcal{H}^s$  of the spin Toda lattice on  $T\mathfrak{h} \times \mathcal{O}_{red}$  is given by*

$$\mathcal{H}_0^s(x, p) = \frac{1}{2} \sum_i p_i^2 - \sum_{\alpha \in \pi'} c_{\alpha} e^{-\alpha(x)} \quad (5.39)$$

where  $c_{\alpha} = s_{-\alpha}$  is a constant. Thus the Hamiltonian equations of motion generated by  $\mathcal{H}_0^s$  are:

$$\begin{aligned} \dot{x} &= p, \\ \dot{p} &= - \sum_{\alpha \in \pi'} c_{\alpha} e^{-\alpha(x)} H_{\alpha}. \end{aligned} \quad (5.40)$$

$\square$

Hence by reduction, we obtain a family of Toda lattices parametrized by subsets  $\pi'$  of  $\pi$ .

## 6. Solution of the hyperbolic spin Calogero-Moser systems and the spin Toda lattices.

(a) *The hyperbolic spin Calogero-Moser systems.*

We begin by solving the equation

$$\begin{aligned} & \frac{d}{dt}(q, 0, L(q, p, \xi)) \\ &= (p, 0, [L(q, p, \xi), R(q) L(q, p, \xi)]). \end{aligned} \quad (6.1)$$

for the hyperbolic spin Calogero-Moser system which we obtain from Proposition 5.4 by restricting to the invariant manifold  $J^{-1}(0)$ . As the reader will see, this will lead us eventually to the solution of the associated integrable model, whose equations are given in Proposition 5.6.

In order to set up the factorization problem properly, it is necessary to have precise knowledge of the Lie algebroids and Lie groupoids involved. Let us begin to describe these objects. Let  $\mathfrak{b}^- = \mathfrak{h} + \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$  and  $\mathfrak{b}^+ = \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$  be opposing Borel subalgebras of  $\mathfrak{g}$ . From the definition of  $R$  in (5.1), we have

$$\begin{aligned} R^\pm(q)\xi &= \pm \frac{1}{2} \sum_i \xi_i x_i - \frac{1}{2} \sum_{\alpha \in \langle \pi' \rangle} \frac{e^{\mp \frac{1}{2}\alpha(q)}}{\sinh \frac{1}{2}\alpha(q)} \xi_\alpha e_\alpha \\ &\quad \pm \sum_{\alpha \in \overline{\pi'}^\mp} \xi_\alpha e_\alpha. \end{aligned} \quad (6.2)$$

Therefore,  $R^+(q)\xi$  is in the parabolic subalgebra

$$\mathfrak{p}_{\pi'}^- = \mathfrak{b}^- + \sum_{\alpha \in \langle \pi' \rangle^+} \mathfrak{g}_\alpha \quad (6.3)$$

containing  $\mathfrak{b}^-$ , while  $R^-(q)\xi$  is in the parabolic subalgebra

$$\mathfrak{p}_{\pi'}^+ = \mathfrak{b}^+ + \sum_{\alpha \in \langle \pi' \rangle^-} \mathfrak{g}_\alpha \quad (6.4)$$

containing  $\mathfrak{b}^+$ . Now, recall that we have the standard decompositions [Kn]

$$\mathfrak{p}_{\pi'}^\pm = \mathfrak{g}_{\pi'}^\pm + \mathfrak{n}_{\pi'}^\pm \quad (6.5)$$

where

$$\mathfrak{g}_{\pi'} = \mathfrak{h} + \sum_{\alpha \in \langle \pi' \rangle} \mathfrak{g}_\alpha \quad (6.6)$$

is the Levi factor of  $\mathfrak{p}_{\pi'}^{\pm}$ , and

$$\mathfrak{n}_{\pi'}^{\pm} = \sum_{\alpha \in \bar{\pi}'^{\pm}} \mathfrak{g}_{\alpha} \quad (6.7)$$

are the nilpotent radicals. Moreover, we have the identity

$$\mathfrak{g} = \mathfrak{n}_{\pi'}^{-} + \mathfrak{g}_{\pi'} + \mathfrak{n}_{\pi'}^{+}. \quad (6.8)$$

Let  $P_{\pi'}^{\pm}$ ,  $G_{\pi'}$  and  $N_{\pi'}^{\pm}$  be the simply-connected Lie subgroups of  $G$  with corresponding Lie subalgebras  $\mathfrak{p}_{\pi'}^{\pm}$ ,  $\mathfrak{g}_{\pi'}$  and  $\mathfrak{n}_{\pi'}^{\pm}$ . Then we have

$$P_{\pi'}^{\pm} = N_{\pi'}^{\pm} G_{\pi'} \quad (6.9)$$

and the submanifold

$$G_0 = N_{\pi'}^{-} G_{\pi'} N_{\pi'}^{+} \quad (6.10)$$

is an open dense subset of  $G$ .

From the explicit expression for  $R^{+}$  in (6.2) and the definition of  $\mathcal{R}^{+}$ , we find that

$$Im\mathcal{R}^{+} = \bigcup_{q \in U} \{0_q\} \times \mathfrak{p}_{\pi'}^{-} \times \mathfrak{h}. \quad (6.11)$$

Similarly, we have

$$Im\mathcal{R}^{-} = \bigcup_{q \in U} \{0_q\} \times \mathfrak{p}_{\pi'}^{+} \times \mathfrak{h}. \quad (6.12)$$

Therefore, the unique source-simply connected Lie groupoids of  $Im\mathcal{R}^{\pm}$  are given by

$$\Gamma_{\pm} = U \times P_{\pi'}^{\mp} \times U. \quad (6.13)$$

We next describe the ideals  $\mathcal{I}^{\pm}$  of (4.7) for the case under consideration.

**Lemma 6.1.**  $\mathcal{I}^{\pm} = \bigcup_{q \in U} \{0_q\} \times \mathfrak{n}_{\pi'}^{\mp} \times \{0\}$ .

*Proof.* We shall give the proof for  $\mathcal{I}^{-}$ . Suppose  $(0_q, X, 0) \in \mathcal{I}^{-}$ , then there exists  $Z \in \mathfrak{h}$  such that  $\mathcal{R}^{+}(0_q, X, Z) = 0$ . Equivalently, we have  $-\iota Z + R^{+}(q)X = 0$  and  $\Pi_{\mathfrak{h}}X = 0$ . But from the explicit expression for  $R^{+}$  in (6.1), we easily find that  $X_{\alpha} = 0$  for  $\alpha \in \Delta^{-} \cup <\pi'>^{+}$ . This shows that  $X \in \mathfrak{n}_{\pi'}^{+}$ . The converse is clear by reversing the steps in the above argument.  $\square$

From this lemma, it follows that

$$\begin{aligned} Im\mathcal{R}^{+}/\mathcal{I}^{+} &= \bigcup_{q \in U} \{0_q\} \times (\mathfrak{p}_{\pi'}^{-}/\mathfrak{n}_{\pi'}^{-}) \times \mathfrak{h} \\ &\simeq \bigcup_{q \in U} \{0_q\} \times \mathfrak{g}_{\pi'} \times \mathfrak{h} \end{aligned} \quad (6.14)$$

where the identification map is given by

$$(0_q, X + \mathfrak{n}_{\pi'}^-, Z) \mapsto (0_q, \Pi_{\mathfrak{g}_{\pi'}}^-, X, Z). \quad (6.15)$$

Here,  $\Pi_{\mathfrak{g}_{\pi'}}^-$  is the projection onto  $\mathfrak{g}_{\pi'}$  relative to the direct sum decomposition  $\mathfrak{p}_{\pi'}^- = \mathfrak{g}_{\pi'} + \mathfrak{n}_{\pi'}^-$ . Similarly,

$$\begin{aligned} \text{Im}\mathcal{R}^-/\mathcal{I}^- &= \bigcup_{q \in U} \{0_q\} \times (\mathfrak{p}_{\pi'}^+/\mathfrak{n}_{\pi'}^+) \times \mathfrak{h} \\ &\simeq \bigcup_{q \in U} \{0_q\} \times \mathfrak{g}_{\pi'} \times \mathfrak{h}. \end{aligned} \quad (6.16)$$

This time, the identification is given by the map

$$(0_q, X + \mathfrak{n}_{\pi'}^+, Z) \mapsto (0_q, \Pi_{\mathfrak{g}_{\pi'}}^+, X, Z). \quad (6.17)$$

and  $\Pi_{\mathfrak{g}_{\pi'}}^+$  is the projection onto  $\mathfrak{g}_{\pi'}$  relative to  $\mathfrak{p}_{\pi'}^+ = \mathfrak{g}_{\pi'} + \mathfrak{n}_{\pi'}^+$ .

**Proposition 6.2.** *The isomorphism  $\theta : \text{Im}\mathcal{R}^+/\mathcal{I}^+ \longrightarrow \text{Im}\mathcal{R}^-/\mathcal{I}^-$  defined in Proposition 4.4 is given by*

$$\theta(0_q, \Pi_{\mathfrak{g}_{\pi'}}^-, X, Z) = (0_q, -\iota Z + \text{Ad}_{e^q} \Pi_{\mathfrak{g}_{\pi'}}^-, X, Z) \quad (6.18)$$

for all  $q \in U$ ,  $X \in \mathfrak{p}_{\pi'}^-$  and  $Z \in \mathfrak{h}$ .

*Proof.* From the expression for  $R^\pm(q)\xi$ , we have

$$\begin{aligned} &\theta(0_q, -\iota Z' + \frac{1}{2}\Pi_{\mathfrak{h}}\xi - \frac{1}{2} \sum_{\alpha \in \langle \pi' \rangle} \frac{e^{-\frac{1}{2}\alpha(q)}}{\sinh \frac{1}{2}\alpha(q)} \xi_\alpha e_\alpha, \Pi_{\mathfrak{h}}\xi) \\ &= (0_q, -\iota Z' - \frac{1}{2}\Pi_{\mathfrak{h}}\xi - \frac{1}{2} \sum_{\alpha \in \langle \pi' \rangle} \frac{e^{\frac{1}{2}\alpha(q)}}{\sinh \frac{1}{2}\alpha(q)} \xi_\alpha e_\alpha, \Pi_{\mathfrak{h}}\xi). \end{aligned}$$

The assertion then follows from the identity  $\text{Ad}_{e^q} e_\alpha = e^{\alpha(q)} e_\alpha$ .  $\square$

We shall make the natural identifications  $N_{\pi'}^\pm \backslash P_{\pi'}^\pm \simeq G_{\pi'}$  using the relation in (6.9) in what follows.

**Corollary 6.3.** *The isomorphism  $\theta$  can be lifted up to a Lie groupoid isomorphism*

$$\begin{aligned} \Theta : U \times (N_{\pi'}^- \backslash P_{\pi'}^-) \times U &\longrightarrow U \times (N_{\pi'}^+ \backslash P_{\pi'}^+) \times U \\ (u, \lambda^-(g), v) &\mapsto (u, e^u \lambda^-(g) e^{-v}, v) \end{aligned} \quad (6.19)$$

where for  $g \in P_{\pi'}^-$ ,  $\lambda^-(g) \in G_{\pi'}$  denotes the factor in the unique factorization  $g = \nu^-(g) \lambda^-(g)$ ,  $\nu^-(g) \in N_{\pi'}^-$ .

*Proof.* This is straightforward verification.  $\square$

We are now ready to solve Eqn.(6.1). To do so, we have to solve the factorization problem

$$\exp\{t(0, 0, L(q_0, p_0, \xi_0))\}(q_0) = \gamma_+(t) \gamma_-(t)^{-1} \quad (6.20)$$

for  $(\gamma_+(t), \gamma_-(t)) = ((q_0, k_+(t), q(t)), (q_0, k_-(t), q(t))) \in \text{Im}(\mathcal{R}^+, \mathcal{R}^-)$  satisfying the condition in (4.19), where  $(q_0, p_0, \xi_0) \in J^{-1}(0) = TU \times (\mathcal{U} \cap \mathfrak{h}^\perp)$  is the initial value of  $(q, p, \xi)$ . We shall use the following notation (analogous to what we did in Corollary 6.3): for  $g \in P_{\pi'}^+$ ,  $\nu^+(g) \in N_{\pi'}^+$ ,  $\lambda^+(g) \in G_{\pi'}$  will denote the factors in the unique factorization  $g = \nu^+(g)\lambda^+(g)$ .

With the notation above, we have  $k_\pm(t) \in P_{\pi'}^\mp$ . Therefore, the relation  $e^{tL(q_0, p_0, \xi_0)} = k_+(t)k_-(t)^{-1}$  (which follows from (6.20)) can be rewritten in the form

$$e^{tL(q_0, p_0, \xi_0)} = \nu^-(k_+(t))\lambda^-(k_+(t))\lambda^+(k_-(t))^{-1}\nu^+(k_-(t))^{-1}.$$

But from Theorem 4.5 (b) and Corollary 6.3, we have  $\lambda^+(k_-(t)) = e^{q_0}\lambda^-(k_+(t))e^{-q(t)}$ . Hence it follows that

$$\begin{aligned} & e^{tL(q_0, p_0, \xi_0)} \\ &= \nu^-(k_+(t))(\lambda^-(k_+(t))e^{q(t)}\lambda^-(k_+(t))^{-1}e^{-q_0})\nu^+(k_-(t))^{-1}. \end{aligned} \quad (6.21)$$

where the middle factor is in  $G_{\pi'}$  and  $\nu^\mp(k_\pm(t)) \in N_{\pi'}^\mp$ . We shall obtain the factors  $\nu^\mp(k_\pm(t))$ ,  $\lambda^-(k_+(t))$  and  $q(t)$  in several steps. First of all, from the fact that  $e^{tL(q_0, p_0, \xi_0)} \in P_{\pi'}^+$ , we can find (as a consequence of (6.9)) unique  $g(t) \in G_{\pi'}$ ,  $n_+(t) \in N_{\pi'}^+$  satisfying  $n_+(0) = g(0) = 1$  such that

$$e^{tL(q_0, p_0, \xi_0)} = g(t)n_+(t)^{-1}. \quad (6.22)$$

By comparing (6.21) and (6.22), we obtain

$$\nu^-(k_+(t)) = 1, \quad \nu^+(k_-(t)) = n_+(t). \quad (6.23)$$

Hence (6.21) reduces to the factorization problem

$$g(t)e^{q_0} = \lambda^-(k_+(t))e^{q(t)}\lambda^-(k_+(t))^{-1}. \quad (6.24)$$

Since  $G_{\pi'}$  is a reductive Lie group, we can find (at least for small values of  $t$ )  $x(t) \in G_{\pi'}$  (unique up to transformations  $x(t) \rightarrow x(t)\delta(t)$ , where  $\delta(t) \in H$ ) and unique  $d(t) \in H$  such that

$$g(t)e^{q_0} = x(t)d(t)x(t)^{-1} \quad (6.25)$$

with  $x(0) = 1$ ,  $d(0) = e^{q_0}$ . This uniquely determines  $q(t)$  via the formula

$$q(t) = \log d(t). \quad (6.26)$$

On the other hand, let us fix one such  $x(t)$ . We shall seek  $\lambda^-(k_+(t))$  in the form

$$\lambda^-(k_+(t)) = x(t)b(t), \quad b(t) \in H. \quad (6.27)$$

To determine  $b(t)$ , we impose the condition in (4.19 a). After some calculations, we find that  $b(t)$  satisfies the equation

$$\dot{b}(t) = T_e l_{b(t)} \left( \frac{1}{2} \dot{q}(t) - \Pi_{\mathfrak{h}}(T_{x(t)} l_{x(t)^{-1}} \dot{x}(t)) \right) \quad (6.28)$$

with  $b(0) = 1$ . Solving the equation explicitly, we find that

$$\lambda^-(k_+(t)) = x(t) \exp \left\{ \frac{1}{2} (q(t) - q_0) - \int_0^t \Pi_{\mathfrak{h}}(T_{x(\tau)} l_{x(\tau)^{-1}} \dot{x}(\tau)) d\tau \right\}. \quad (6.29)$$

Combining (6.23) and (6.29), we finally have

$$k_+(t) = x(t) \exp \left\{ \frac{1}{2} (q(t) - q_0) - \int_0^t \Pi_{\mathfrak{h}}(T_{x(\tau)} l_{x(\tau)^{-1}} \dot{x}(\tau)) d\tau \right\}. \quad (6.30)$$

Hence we can write down the solution of Eqn.(6.1) by using (4.20). Note, however, we cannot determine  $\xi(t)$  from the solution for  $L$  as the expression for  $L$  does not involve  $\xi_\alpha$  for  $\alpha \in \bar{\pi}'^-$ . The solution of Eqn.(5.7) on  $J^{-1}(0)$  is given in the following.

**Theorem 6.4.** *Let  $(q_0, p_0, \xi_0) \in J^{-1}(0) = TU \times (\mathcal{U} \cap \mathfrak{h}^\perp)$ . Then the Hamiltonian flow on  $J^{-1}(0)$  generated by*

$$\begin{aligned} \mathcal{H}(q, p, \xi) = & \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_i \xi_i^2 + \frac{1}{2} \sum_i p_i \xi_i \\ & - \frac{1}{8} \sum_{\alpha \in \langle \pi' \rangle} \frac{\xi_\alpha \xi_{-\alpha}}{\sinh^2 \frac{1}{2} \alpha(q)} \end{aligned}$$

with initial condition  $(q(0), p(0), \xi(0)) = (q_0, p_0, \xi_0)$  is given by

$$\begin{aligned} q(t) &= \log d(t), \\ \xi(t) &= Ad_{k_+(t)^{-1}} \xi_0, \\ p(t) &= Ad_{k_\pm(t)^{-1}} L(q_0, p_0, \xi_0) - \frac{1}{2} \sum_{\alpha \in \langle \pi' \rangle} \frac{e^{\frac{1}{2} \alpha(q(t))}}{\sinh \frac{1}{2} \alpha(q(t))} \xi_\alpha(t) e_\alpha \\ &\quad - \sum_{\alpha \in \bar{\pi}'^+} \xi_\alpha(t) e_\alpha \end{aligned} \quad (6.31)$$

where  $d(t)$ ,  $k_{\pm}(t)$  are constructed from the above procedure and in the expression for  $p(t)$ , the quantities  $q(t)$  and  $\xi(t)$  which appear on the right hand side are given by the formulas above that expression.

*Proof.* We first show  $\xi(t) = Ad_{k_+(t)^{-1}}\xi_0$  solves the equation  $\dot{\xi} = [\xi, R^+(q)L(q, p, \xi)]$  in Proposition 5.4. To do so, we differentiate the expression for  $\xi(t)$ , this gives

$$\dot{\xi}(t) = [T_{k_+(t)^{-1}}r_{k_+(t)}\frac{d}{dt}k_+(t)^{-1}, \xi(t)].$$

But

$$\begin{aligned} & T_{k_+(t)^{-1}}r_{k_+(t)}\frac{d}{dt}k_+(t)^{-1} \\ &= -T_{k_+(t)}l_{k_+(t)^{-1}}\dot{k}_+(t) \\ &= -R^+(q(t))L(q(t), p(t), \xi(t)) \end{aligned}$$

from the argument in Theorem 4.7. Hence the claim. To get the formula for  $p(t)$ , we simply equate the following two expressions for  $L((q(t), p(t), \xi(t)))$ , namely,  $L(q(t), p(t), \xi(t)) = Ad_{k_{\pm}(t)^{-1}}L(q_0, p_0, \xi_0)$  and

$$\begin{aligned} L(q(t), p(t), \xi(t)) &= p(t) + \frac{1}{2} \sum_{\alpha \in \langle \pi' \rangle} \frac{e^{\frac{1}{2}\alpha(q(t))}}{\sinh \frac{1}{2}\alpha(q(t))} \xi_{\alpha}(t) e_{\alpha} \\ &+ \sum_{\alpha \in \overline{\pi'}^+} \xi_{\alpha}(t) e_{\alpha}. \end{aligned}$$

This completes the proof.  $\square$

We next turn to the solution of the associated integrable model on  $TU \times \mathfrak{g}_{red}$  with Hamiltonian  $\mathcal{H}_0(q, p, s) = \frac{1}{2} \sum_i p_i^2 - \frac{1}{4} \sum_{\alpha \in \langle \pi' \rangle} \frac{s_{\alpha} s_{-\alpha}}{\sinh^2 \frac{1}{2}\alpha(q)}$  and with equations of motion given in Proposition 5.6.

**Corollary 6.5.** *Let  $(q_0, p_0, s_0) \in TU \times \mathfrak{g}_{red}$  and suppose  $s_0 = Ad_{g(\xi_0)^{-1}}\xi_0$  where  $\xi_0 \in \mathcal{U} \cap \mathfrak{h}^{\perp}$ . Then the Hamiltonian flow generated by  $\mathcal{H}_0$  with initial condition  $(q(0), p(0), s(0)) = (q_0, p_0, s_0)$  is given by*

$$\begin{aligned} q(t) &= \log d(t), \\ s(t) &= Ad_{(\tilde{k}_+(t)g(Ad_{\tilde{k}_+(t)^{-1}}s_0))^{-1}}s_0, \\ p(t) &= Ad_{(\tilde{k}_+(t)g(Ad_{\tilde{k}_+(t)^{-1}}s_0))^{-1}}(p_0 - R^-(q_0)s_0) + R^-(q(t))s(t). \end{aligned} \tag{6.32}$$

where  $\tilde{k}_+(t) = g(\xi_0)^{-1}k_+(t)g(\xi_0)$  and  $k_+(t)$ ,  $d(t)$  are as in Theorem 6.4.

*Proof.* We shall obtain the Hamiltonian flow generated by  $\mathcal{H}_0$  by reduction. Using the relation  $\phi_t^{red} \circ \pi_0 = \pi_0 \circ \phi_t \circ i_0$  from Theorem 3.2 (c), we have  $\phi_t^{red}(q_0, p_0, s_0) = (q(t), p(t), Ad_{g(\xi(t))^{-1}} \xi(t))$  where  $q(t), p(t), \xi(t)$  are given by the expressions in Theorem 6.4. Thus

$$\begin{aligned} s(t) &= Ad_{g(\xi(t))^{-1}} \xi(t) \\ &= Ad_{g(Ad_{k_+(t)^{-1}} \xi_0)^{-1} Ad_{k_+(t)^{-1} g(\xi_0)} s_0} \\ &= Ad_{(\tilde{k}_+(t) g(Ad_{\tilde{k}_+(t)^{-1}} s_0))^{-1} s_0} \end{aligned}$$

where we have used the  $H$ -equivariance of the map  $g$ . To express  $p(t)$  in the desired form, introduce the gauge transformation of  $L$ :  $\tilde{L}(q, p, s) = Ad_{g(\xi)^{-1}} L(q, p, \xi) = p - R^-(q)s$ , where as before,  $s = Ad_{g(\xi)^{-1}} \xi$ . Then

$$\begin{aligned} \tilde{L}(q(t), p(t), s(t)) &= Ad_{g(\xi(t))^{-1}} Ad_{k_+(t)^{-1}} L(q_0, p_0, \xi_0) \\ &= Ad_{(\tilde{k}_+(t) g(Ad_{\tilde{k}_+(t)^{-1}} s_0))^{-1}} (p_0 - R^-(q_0) s_0). \end{aligned}$$

But  $\tilde{L}(q(t), p(t), s(t))$  is also equal to  $p(t) - R^-(q(t))s(t)$ . By equating the two expressions, we obtain the desired expression for  $p(t)$ , as claimed.  $\square$

**Remark 6.6** (a) It is easy to show that the element  $\tilde{k}_+(t) = g(\xi_0)^{-1} k_+(t) g(\xi_0)$  depends only on  $s_0$ , and not on the particular element  $\xi_0 \in \mathcal{U} \cap \mathfrak{h}^\perp$  for which  $Ad_{g(\xi_0)^{-1}} \xi_0 = s_0$ . Indeed, from the factorization  $e^{L(q_0, p_0, \xi_0)} = k_+(t) k_-(t)^{-1}$ , we see that if we replace  $\xi_0$  by  $Ad_h \xi_0$ ,  $h \in H$ , then the factors  $k_\pm(t)$  will be replaced by  $h k_\pm(t) h^{-1}$ . As  $g(Ad_h \xi_0) = h g(\xi_0)$ , our assertion easily follows. Note that this is exactly the reason why we have chosen to express  $s(t)$  and  $p(t)$  in the form given in the above Corollary.

(b) In [L1], we introduced a family of hyperbolic spin Ruijsenaars-Schneider models on the coboundary dynamical Poisson groupoids  $(\Gamma = U \times G \times U, \{\cdot, \cdot\}_R)$  associated to the same  $R$ 's which we use here. Recall that these are generated by nonzero multiples of  $H_i = Pr_2^* \chi_i$ ,  $i = 1, \dots, N$ , where  $Pr_2$  in this present case denotes projection onto the second factor of  $\Gamma$ , and  $\chi_1, \dots, \chi_N$  are the characters of the irreducible representations corresponding to the fundamental weights  $\omega_1, \dots, \omega_N$  [St]. If we take the Hamiltonian  $H_i$ , say, then its Hamiltonian flow on the gauge group bundle  $\mathcal{I}\Gamma$  is defined by the equation

$$\begin{aligned} &\frac{d}{dt}(q, g, q) \\ &= \left(-\frac{1}{2} \Pi_{\mathfrak{h}} D\chi_i(g), \frac{1}{2} T_{erg} R(q)(D\chi_i(g)) - \frac{1}{2} T_{elg} R(q)(D\chi_i(g)), -\frac{1}{2} \Pi_{\mathfrak{h}} D\chi_i(g)\right) \end{aligned}$$

where  $D\chi_i(g)$  is the right gradient of  $\chi_i$ . In this case, the factorization problem on  $\Gamma$  (from [L1] and our analysis above) gives

$$\begin{aligned} & e^{-tD\chi_i(g_0)} \\ &= \boldsymbol{\nu}^-(k_+(t))(\boldsymbol{\lambda}^-(k_+(t))e^{q(t)}\boldsymbol{\lambda}^-(k_+(t))^{-1}e^{-q_0})\boldsymbol{\nu}^+(k_-(t))^{-1} \end{aligned}$$

using the same notation as before (of course, the  $k_{\pm}$  here are different from the ones above). If for small  $t$ ,  $n_{\pm}(t) \in N_{\pi'}^{\pm}$ ,  $g(t) \in G_{\pi'}$  are the unique solution of the factorization problem

$$e^{-tD\chi_i(g_0)} = n_-(t)g(t)n_+(t)^{-1}$$

satisfying  $n_{\pm}(0) = g(0) = 1$ , then

$$\boldsymbol{\nu}^-(k_+(t)) = n_-(t), \quad \boldsymbol{\nu}^+(k_-(t)) = n_+(t).$$

Consequently, the factorization problem reduces to

$$g(t)e^{q_0} = \boldsymbol{\lambda}^-(k_+(t))e^{q(t)}\boldsymbol{\lambda}^-(k_+(t))^{-1}$$

and therefore the solution proceeds as before. Finally, we can write down the Hamiltonian flow on  $\mathcal{I}\Gamma$  using the formula from [L1], namely,

$$\begin{aligned} & (q(t), g(t), q(t)) \\ &= (q_0, k_{\pm}(t), q(t))^{-1}(q_0, g_0, q_0)(q_0, k_{\pm}(t), q(t)). \end{aligned}$$

(b) *The spin Toda lattices.*

In this final subsection, we shall discuss the solution of the spin Toda lattices. In this case, we have

$$Im\mathcal{R}^{\pm} = \bigcup_{q \in \mathfrak{h}} \{0_q\} \times \mathfrak{b}^{\mp} \times \mathfrak{h} \quad (6.33)$$

where  $\mathfrak{b}^{\pm}$  are the opposing Borel subalgebra of  $\mathfrak{g}$  introduced at the beginning of the section. Let  $B^{\pm}$  be the simply-connected Lie subgroups of  $G$  which integrate  $\mathfrak{b}^{\pm}$ , then the unique source-simply connected Lie groupoid of  $Im\mathcal{R}^{\pm}$  are given by

$$\Gamma_{\pm} = \mathfrak{h} \times B^{\mp} \times \mathfrak{h}. \quad (6.34)$$

Now, the ideals  $\mathcal{I}^{\pm}$  of (4.7) in the present case are:

$$\mathcal{I}^{\pm} = \bigcup_{q \in \mathfrak{h}} \{0_q\} \times \mathfrak{n}^{\mp} \times \{0\} \quad (6.35)$$

where  $\mathfrak{n}^{\pm} = \sum_{\alpha \in \Delta^{\pm}} \mathfrak{g}_{\alpha}$ . We shall denote by  $N^{\pm}$  the simply-connected Lie subgroups of  $G$  with  $Lie(N^{\pm}) = \mathfrak{n}^{\pm}$ . To cut the story short, we have the following result when we go through the analysis.

**Theorem 6.7.** *Let  $(x_0, p_0, \eta_0) \in T\mathfrak{h} \times \mathfrak{g} \simeq S^*$  and let  $n_{\pm}(t) \in N^{\pm}$ ,  $h(t) \in H$  be the unique solution of the factorization problem*

$$e^{t\mathbf{L}(x_0, p_0, \eta_0)} = n_{-}(t)h(t)n_{+}(t)^{-1} \quad (6.36)$$

*(valid for  $0 \leq t < T$  for some  $T > 0$ ) satisfying  $n_{\pm}(0) = h(0) = 1$ . Then the Hamiltonian flow on  $T\mathfrak{h} \times \mathfrak{g} \simeq S^*$  generated by the Hamiltonian*

$$\begin{aligned} \mathcal{H}^s(x, p, \eta) = & \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_i \eta_i^2 + \frac{1}{2} \sum_i p_i \eta_i \\ & - \sum_{\alpha \in \pi'} \eta_{\alpha} \eta_{-\alpha} e^{-\alpha(x)} \end{aligned}$$

*with initial condition  $(x(0), p(0), \eta(0)) = (x_0, p_0, \eta_0)$  is given by*

$$\begin{aligned} x(t) &= x_0 + \log h(t), \\ \eta(t) &= Ad_{e^{-\frac{1}{2} \log h(t)}} \eta_0, \\ p(t) &= Ad_{k_{\pm}(t)^{-1}} \mathbf{L}(x_0, p_0, \eta_0) - \frac{1}{2} \Pi_{\mathfrak{h}} \eta_0 - \sum_{\alpha \in \pi} e^{-\frac{1}{2} \alpha(\log h(t))} (\eta_0)_{-\alpha} e_{\alpha} \\ &\quad + \sum_{\alpha \in \pi'} e^{-\alpha(x_0 + \frac{1}{2} \log h(t))} (\eta_0)_{-\alpha} e_{\alpha} \end{aligned} \quad (6.37)$$

where

$$k_{\pm}(t) = n_{\mp}(t) e^{\pm \frac{1}{2} \log h(t)}. \quad (6.38)$$

*Proof.* The expression for  $x(t)$  is clear if we write down the expression analogous to (6.21) and compare that with the factorization in (6.36). On the other hand, it is clear from the same expression that  $k_{+}(t) = n_{-}(t)b_{-}(t)$  where  $b_{-}(t) \in H$  is to be determined from the condition given in (4.19). If we spell this out, we find that  $b_{-}(t)$  satisfies the equation

$$\dot{b}_{-}(t) = \frac{1}{2} T_e l_{b_{-}(t)} \dot{x}(t)$$

with  $b_{-}(0) = 1$ . Solving the equation explicitly, we have  $b_{-}(t) = e^{\frac{1}{2}(x(t) - x_0)}$ . Now, in order to solve the equation for  $\eta$  in (5.24), the crucial point to note is that we can rewrite this equation as  $\dot{\eta} = [\eta, \frac{1}{2} \Pi_{\mathfrak{h}} \mathbf{L}(x, p, \eta)]$ . Finally, we can obtain the formula for  $p(t)$  by equating the following two expressions for  $\mathbf{L}(x(t), p(t), \eta(t))$ , namely,  $\mathbf{L}(x(t), p(t), \eta(t)) = Ad_{k_{\pm}(t)^{-1}} \mathbf{L}(x_0, p_0, \eta_0)$  and

$$\mathbf{L}(x(t), p(t), \eta(t)) = p(t) + \frac{1}{2} \Pi_{\mathfrak{h}} \eta_0 + \sum_{\alpha \in \pi} \eta_{\alpha}(t) e_{\alpha} - \sum_{\alpha \in \pi'} e^{-\alpha(x(t))} \eta_{-\alpha}(t) e_{-\alpha}.$$

This completes the proof.  $\square$

The solution of the family of Toda lattices in Corollary 5.12 is now straightforward. We shall leave the easy details to the reader.

## Appendix

*Proof of Lemma 4.1.*

From the definition of  $\mathcal{R}$  and the expression for  $[\cdot, \cdot]_{A\Gamma}$ ,  $[\cdot, \cdot, \cdot]_{A^*\Gamma}$ , we have

$$\begin{aligned} & [\mathcal{R}(0, A, Z), \mathcal{R}(0, A', Z')]_{A\Gamma}(q) - \mathcal{R}[(0, A, Z), (0, A', Z')]_{A^*\Gamma}(q) \\ &= (0_q, \mathcal{A}, \mathcal{Z}) \end{aligned}$$

where (after the obvious cancellations)

$$\begin{aligned} \mathcal{A} = & [R(q)A(q), R(q)A'(q)] + \langle dR(q)(\cdot)A(q), A'(q) \rangle + \left\{ R(q)ad_{R(q)A(q)}^* A'(q) \right. \\ & + dR(q)\iota^* A(q)(A'(q)) - dR(q)ad_{Z(q)}^* q(A'(q)) - [Z(q), R(q)A'(q)] \\ & \left. - R(q)ad_{Z(q)}^* A'(q) - (A \leftrightarrow A', Z \leftrightarrow Z') \right\} \end{aligned}$$

and

$$\mathcal{Z} = \iota^* ad_{R(q)A(q)}^* A'(q) - \iota^* ad_{R(q)A'(q)}^* A(q) + ad_{\langle dR(q)(\cdot)A(q), A'(q) \rangle}^* q.$$

Since  $R$  is  $H$ -equivariant, we can show that  $\mathcal{Z} = 0$  and

$$dR(q)ad_{Z(q)}^* q(A'(q)) + [Z(q), R(q)A'(q)] + R(q)ad_{Z(q)}^* A'(q) = 0.$$

Therefore, the expression for  $\mathcal{A}$  becomes

$$\begin{aligned} \mathcal{A} = & [R(q)A(q), R(q)A'(q)] + \langle dR(q)(\cdot)A(q), A'(q) \rangle \\ & + R(q)(ad_{R(q)A(q)}^* A'(q) - ad_{R(q)A'(q)}^* A(q)) \\ & + dR(q)\iota^* A(q)(A'(q)) - dR(q)\iota^* A'(q)(A(q)) \\ = & - [K(A(q)), K(A'(q))], \end{aligned}$$

as desired.

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L.-C. LI, DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY UNIVERSITY PARK, PA 16802, USA

*E-mail address:* luenli@math.psu.edu